Tests and related questions

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Introduction 1/2

Taking into account the methology of applied statistics, the first task of the researcher is to define the topic of interest and ask the question(s) of interest. Let us give three examples in different fields.

- Field : social sciences
  Topic of interest : the height of bielorussian students

Question of interest : are the students of Minsk university taller than other students in Bielorussia ?

- Field : medical sciences
  Topic of interest : the therapy of a specific disease

Question of interest : is treatment A more efficient than treatment B ?

Introduction 2/2

- Field : economical sciences
  Topic of interest : the economical growth

Question of interest : does the rate of the added valued tax modify the growth ?

The topic and the question(s) of interest are given in the protocol. It has to be done at the beginning of the work.This calendar is often not followed by users of statistics. It can lead to big errors. To answer to the question of interest (i.e. to take a decision), it requires to have some knowledges on the theory of statistical tests. In section 2, we recall the general principles of the theory of statistical tests. In sections 3 and 4, we focus our attention on the power of a test.

Framework

Consider a one-parameter model. Let θ be the unknown parameter, $\theta \in \Theta$. We have two hypotheses $H0$ and $H1$, that correspond to different values of θ :

$$H0 : \theta \in \Theta_0 \qquad \textit{versus} \qquad H1 : \theta \in \Theta_1,$$

where

$$\Theta_0 \cap \Theta_1 = \emptyset \qquad \textit{versus} \qquad \Theta_0 \cup \Theta_1 = \Theta.$$

The purpose of the theory is to discover if the unknown parameter θ is in Θ_0 or in Θ_1.

Test function

We have a sample $X_1, .., X_n$ and its observations $x_1, .., x_n$.

Definition

A test function is a random function $\phi(X_1, .., X_n)$ taking its values in {0,1}. We select

$$H0 \qquad when \qquad \phi = 0, H1 \qquad when \qquad \phi = 1.$$

Errors

We decide correctly if $\phi = 0$ when $\theta \in \Theta_0$ or if $\phi = 1$ when $\theta \in \Theta_1$.
We do not decide correctly if $\phi = 1$ when $\theta \in \Theta_0$ or if $\phi = 0$ when $\theta \in \Theta_1$.

Two types errors of errors :

- $\theta \in \Theta_0$, $H0$ is true and $\phi = 1$ (i.e. rejection of $H0$). This is the error of the first kind.
- $\theta \in \Theta_1$, $H0$ is false and $\phi = 0$ (i.e no rejection of $H0$). This is the error of the second kind.
  Remark. No rejection $\neq$ acceptance

The mathematical scheme 1/2

Consider the function

$$\theta \longrightarrow E_\theta(\phi).$$

We have

$$E_\theta(\phi) = P(\phi = 1),$$

and therefore

$$E_\theta(1 - \phi) = P(\phi = 0).$$

The mathematical scheme 2/2

If $\theta \in \Theta_0$, then $E_\theta(\phi)$ is the error of the first kind.
If $\theta \in \Theta_1$, then $E_\theta(1-\phi)$ is the error of the second kind.

Definition

If $\theta \in \Theta_1$, then the power of a test $\beta(\theta)$ is the quantity defined as follows :

$$\beta(\theta) = 1 - E_\theta(1-\phi) = E_\theta(\phi).$$

Remark. The value of the power $\beta(\theta)$ depends on the value of $\theta \in \Theta_1$.

Purpose

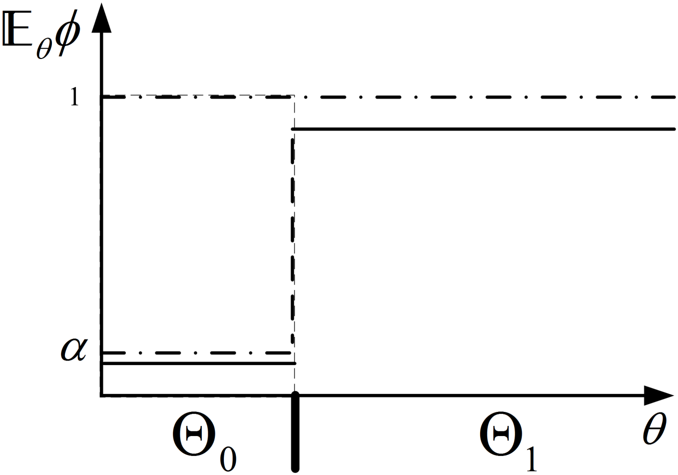
Purpose. Select ϕ such that $E_\theta(\phi)$ is small when $\theta \in \Theta_0$ and $\beta(\theta)$ is big when $\theta \in \Theta_1$.

Remark. Keep in mind that

$$\forall\, \theta \in \Theta \qquad 0 \leq E_\theta(\phi) \leq 1.$$

The ideal situation would be that $E_\theta(\phi)$ is very small when $\theta \in \Theta_0$ and close to 1 when $\theta \in \Theta_1$.

Graphic 1



New purpose

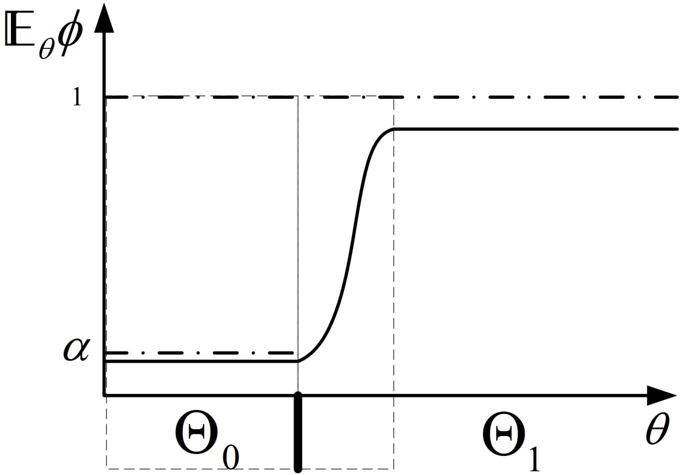
But this is not possible in general. Hence the idea is to privilege one of the two hypotheses : it will always be the hypothesis $H0$. We insist on the fact that the two hypotheses do not play the same role.

New purpose. Let fix the level of significance α, $0 < \alpha < 1$ (in general $\alpha = 0.05$). We make the assumption that the error of the first kind is bounded by α, i.e. we limit ourselves to the test functions ϕ such that :

$$\forall\, \theta \in \Theta_0 \qquad E_\theta(\phi) \leq \alpha.$$

Among these tests, we select those such that $\beta(\theta)$ is close to 1 when $\theta \in \Theta_1$ and θ is not to close from Θ_0.

Graphic 2



Results

The theory of statistical tests gives mechanic answers in most classical cases (estimation of a mean, of a percent, of a variance,...).

Remark. the size of the sample can modify the answer (real law, limit law).

Results of a statistical test

- Rejection of $H0$: the work is complete
- No rejection of $H0$: to accept $H0$, you must compute the power of the test. Among others, this depends on the sample (size, variance,...).

Framework

First, we limit ourselves to the following test ($n \geq 30$,level of significance α, reference value m_0)
We have a sample $X_1, .., X_n$ and its observations $x_1, .., x_n$.
Additional assumption. Under $H0$ and $H1$, σ^2takes the same known value.
Under $H0$, the central limit theorem implies that the sequence

$$\sqrt{n}\frac{\bar{X}_n - m_0}{\sigma}$$

converges in law to the $N(0, 1)$ random variable Y.

Expression of the power

The expression of the power is given by :

$$\beta = P_{H_1}\left(\mid \sqrt{n}\,\frac{\bar{X}_n - m_0}{\sigma} \mid \geq \; u_{\alpha/2} \right)$$

$$= P\left(\mid \sqrt{n}\,\frac{\bar{X}_n - m_0}{\sigma} \mid \geq \; u_{\alpha/2} \mid H_1 \right),$$

where

$$P\left(Y \geq u_{\alpha/2}\right) = \frac{\alpha}{2}.$$

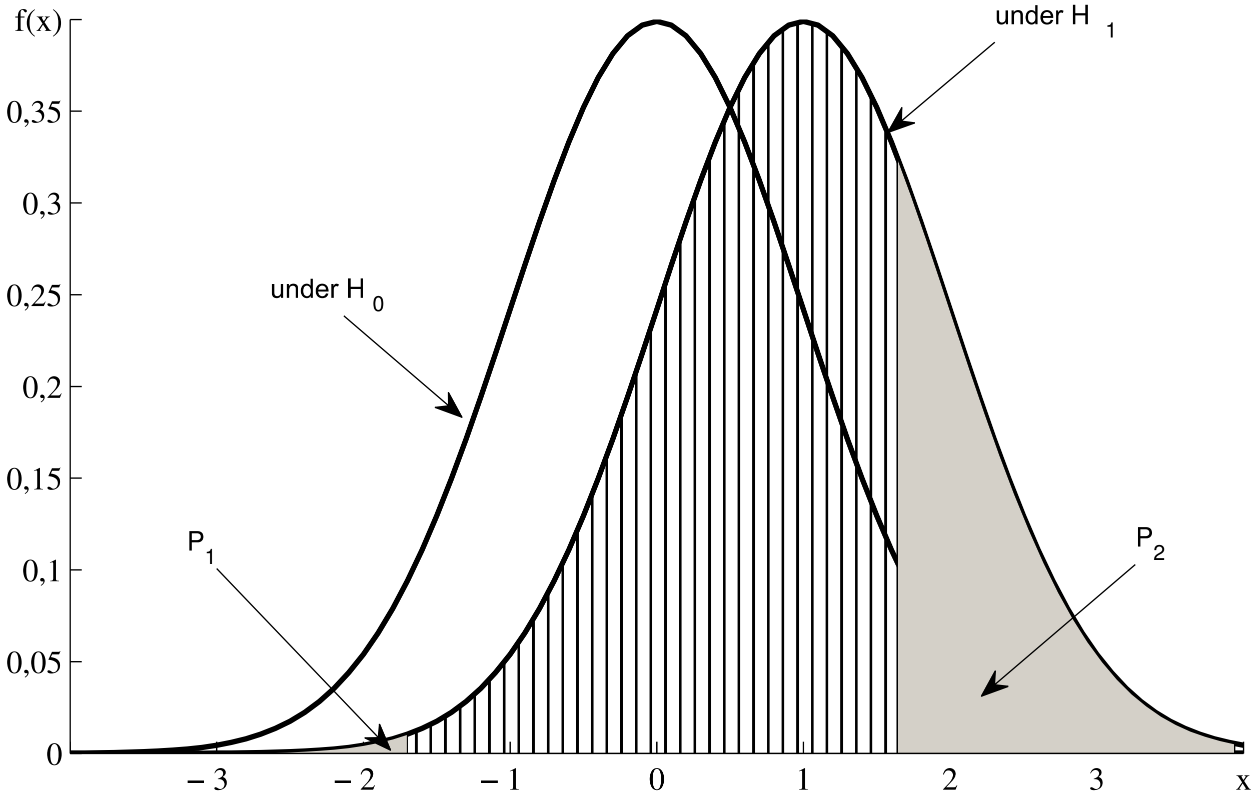
Under $H1$, the central limit theorem implies that the sequence

$$\sqrt{n}\frac{\bar{X}_n - m_1}{\sigma}$$

converges in law to the $N(0,1)$ random variable Y.

Lemma

The law under H1,of $\sqrt{n}\frac{\bar{X}_n - m_0}{\sigma}$ is "approximatively" a normal law with expectation $\sqrt{n}\frac{m_1 - m_0}{\sigma}$ and unit variance.



f(x)
0,35
0,3
0,25
0,2
0,15
0,1
0,05
0
under H 0
under H 1
P 1
P 2
– 3
– 2
– 1
0
1
2
3
x

General formula

General formula to calculate the power

Lemma

$$u_{\alpha/2} - u_{\beta} = \frac{|\, m_1 - m_0 \,|}{\sqrt{\sigma^2 / n}} := \frac{\Delta}{\sqrt{\sigma^2 / n}}.$$

Comments. The power is

- a nondecreasing function of Δ
- a nonincreasing function of σ^2
- a nondecreasing function of n.

Example

Consider the test

$$H0 : m = 2 \quad versus \quad H1 : m(= m_1) \neq 2.$$

Take

$$n = 100, \alpha = 0.05, \sigma^2 = 100.$$

When $H1 : m\,(= m_1) = 3,$

$$\beta = 0.17$$

When $H1 : m\,(= m_1) = 5,$

$$\beta = 0.85$$

Proof.

Set Y a $N(0,1)$ random variable. We have

$$\beta = P\left(Y \geq u_{\alpha/2} - \sqrt{n}\frac{m_1 - m_0}{\sigma}\right)$$

$$+P\left(Y \leq u_{\alpha/2} - \sqrt{n}\frac{m_1 - m_0}{\sigma}\right).$$

To complete the proof, it suffices to distinguish the two following cases :

$$m_1 > m_0 \ \ and \ m_1 < m_0.$$

□

Size of a sample

How to select the size of a sample ? Use

$$n = \frac{\sigma^2}{\Delta^2} \left(u_{\alpha/2} - u_\beta\right)^2 .$$

Example

Consider the test

$$H0 : m = 0 \quad versus \quad H1 : m \neq 0.$$

Take

$$\alpha = 0.05, \sigma^2 = 25, \Delta \geq 1.$$

When $\beta = 0.95$,

$$n = 325$$

When $\beta = 0.8$,

$$n = 196$$

In practice

- List the background of your field of interest
- Determine the question of interest
- Select the suitable test (keep in mind that $H0$ and $H1$ do not play the same role)
- Write the corresponding formula of the size of a sample
- Fix the different quantities
- $\alpha = 0.05$ in general
- An acceptable value of the power $(\beta = 0.8)$
- A range for the ratio $\left(1 \leq \frac{\sigma^2}{\Delta^2} \leq 2\right)$
- The size n is given by the formula
- For guaranty, add several percents $(5\%$ or $10\%)$ to the value of n

The size of your sample is selected.

Unilateral test

When $H1 : m(= m_1) > m_0$, we get the new formula

$$u_\alpha - u_\beta = \frac{(m_1 - m_0)}{\sqrt{\sigma^2 / n}}.$$

Framework

Consider two samples now.
First sample : size $n_1 \geq 30$, expectation m_1
Second sample : size $n_2 \geq 30$, expectation m_2
Assume thet they are independent.
Consider the following test :

$$H0 : m_0 = m_1 \quad versus \quad H1 : m_0 \neq m_1 .$$

Additional assumption. Under $H0$ and $H1$, the variance σ^2takes the same known value.

The central limit theorem implies that the sequences

$$\sqrt{n_1}\frac{\bar{X}_{n1} - m_1}{\sigma} \text{ and } \sqrt{n_2}\frac{\bar{X}_{n2} - m_2}{\sigma}$$

converge in law to the $N(0,1)$ random variable Y.

Under $H0$, combining the independence with the central limit theorem, we get that the sequence

$$\frac{\bar{X}_{n1} - \bar{X}_{n2}}{\sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

converges in law to the $N(0,1)$ random variable Y.

Expression of the power

The expression of the power is given by :

$$\beta = P_{H_1}\left(\mid \frac{\bar{X}_{n1} - \bar{X}_{n2}}{\sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \mid \geq \ u_{\alpha/2} \right)$$

Under $H1$, the central limit theorem implies that the sequence

$$\frac{\bar{X}_{n1} - m_1 - \bar{X}_{n2} + m_2}{\sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

converges in law to the $N(0,1)$ random variable Y.

Lemma

The law under H1,of $\frac{\bar{X}_{n1} - \bar{X}_{n2}}{\sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$ *is "approximatively" a normal law with expectation* $\frac{m_1 - m_2}{\sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$ *and unit variance.*

General fomula

General formula to calculate the power

Lemma

$$u_{\alpha/2} - u_{\beta} = \frac{\mid m_1 - m_2 \mid}{\sigma \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}.$$

Comments. The power is

- a nondecreasing function of Δ
- a nonincreasing function of σ^2
- a nondecreasing function of $n_1$and n_2.

Size of the samples

How to select the size of a sample ? Use

$$\frac{n_1 n_2}{n_1 + n_2} = \frac{\sigma^2}{\Delta^2} \left(u_{\alpha/2} - u_\beta\right)^2 .$$

This equation has several solutions. To solve it, a classical way consists in setting

$$n_2 = k n_1 ,$$

where k is an integer. In practice $k \in \{1, 2, 3, 4\}$.

Example

Example

Take

$$\alpha = 0.05, \beta = 0.95, \frac{\sigma^2}{\Delta^2} = 5$$

Set $n_2 = kn_1$.We have

$$n_1 = \frac{k+1}{k}\frac{\sigma^2}{\Delta^2}\,\left(u_{\alpha/2} - u_\beta\right)^2 = 64.8\,\frac{k+1}{k}.$$

Therefore

$$k = 1, n_1 = n_2 = 130$$

$$k = 2, n_1 = 98, n_2 = 196$$

$$k = 3, n_1 = 87, n_2 = 261$$

$$k = 10, n_1 = 72, n_2 = 720$$

Thanks for your attention? Questions ?