

## 16.2 Line Integrals

In this section we define an integral that is similar to a single integral except that instead of integrating over an interval  $[a, b]$ , we integrate over a curve  $C$ . Such integrals are called *line integrals*, although “curve integrals” would be better terminology. They were invented in the early 19th century to solve problems involving fluid flow, forces, electricity, and magnetism.

We start with a plane curve  $C$  given by the parametric equations

$$\boxed{1} \quad x = x(t) \quad y = y(t) \quad a \leq t \leq b$$

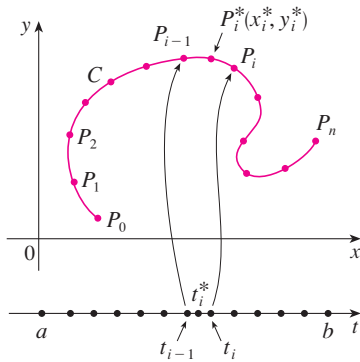


FIGURE 1

or, equivalently, by the vector equation  $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ , and we assume that  $C$  is a smooth curve. [This means that  $\mathbf{r}'$  is continuous and  $\mathbf{r}'(t) \neq \mathbf{0}$ . See Section 13.3.] If we divide the parameter interval  $[a, b]$  into  $n$  subintervals  $[t_{i-1}, t_i]$  of equal width and we let  $x_i = x(t_i)$  and  $y_i = y(t_i)$ , then the corresponding points  $P_i(x_i, y_i)$  divide  $C$  into  $n$  subarcs with lengths  $\Delta s_1, \Delta s_2, \dots, \Delta s_n$ . (See Figure 1.) We choose any point  $P_i^*(x_i^*, y_i^*)$  in the  $i$ th subarc. (This corresponds to a point  $t_i^*$  in  $[t_{i-1}, t_i]$ .) Now if  $f$  is any function of two variables whose domain includes the curve  $C$ , we evaluate  $f$  at the point  $(x_i^*, y_i^*)$ , multiply by the length  $\Delta s_i$  of the subarc, and form the sum

$$\sum_{i=1}^n f(x_i^*, y_i^*) \Delta s_i$$

which is similar to a Riemann sum. Then we take the limit of these sums and make the following definition by analogy with a single integral.

**2 Definition** If  $f$  is defined on a smooth curve  $C$  given by Equations 1, then the **line integral of  $f$  along  $C$**  is

$$\int_C f(x, y) ds = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*, y_i^*) \Delta s_i$$

if this limit exists.

In Section 10.2 we found that the length of  $C$  is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

A similar type of argument can be used to show that if  $f$  is a continuous function, then the limit in Definition 2 always exists and the following formula can be used to evaluate the line integral:

$$\boxed{3} \quad \int_C f(x, y) ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

The value of the line integral does not depend on the parametrization of the curve, provided that the curve is traversed exactly once as  $t$  increases from  $a$  to  $b$ .

The arc length function  $s$  is discussed in Section 13.3.

If  $s(t)$  is the length of  $C$  between  $\mathbf{r}(a)$  and  $\mathbf{r}(t)$ , then

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

So the way to remember Formula 3 is to express everything in terms of the parameter  $t$ : Use the parametric equations to express  $x$  and  $y$  in terms of  $t$  and write  $ds$  as

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

In the special case where  $C$  is the line segment that joins  $(a, 0)$  to  $(b, 0)$ , using  $x$  as the parameter, we can write the parametric equations of  $C$  as follows:  $x = x$ ,  $y = 0$ ,  $a \leq x \leq b$ . Formula 3 then becomes

$$\int_C f(x, y) ds = \int_a^b f(x, 0) dx$$

and so the line integral reduces to an ordinary single integral in this case.

Just as for an ordinary single integral, we can interpret the line integral of a *positive* function as an area. In fact, if  $f(x, y) \geq 0$ ,  $\int_C f(x, y) ds$  represents the area of one side of the “fence” or “curtain” in Figure 2, whose base is  $C$  and whose height above the point  $(x, y)$  is  $f(x, y)$ .

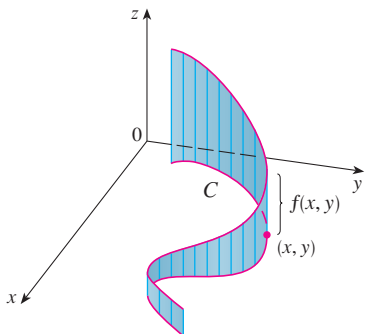


FIGURE 2

**EXAMPLE 1** Evaluate  $\int_C (2 + x^2y) ds$ , where  $C$  is the upper half of the unit circle  $x^2 + y^2 = 1$ .

**SOLUTION** In order to use Formula 3, we first need parametric equations to represent  $C$ . Recall that the unit circle can be parametrized by means of the equations

$$x = \cos t \quad y = \sin t$$

and the upper half of the circle is described by the parameter interval  $0 \leq t \leq \pi$ . (See Figure 3.) Therefore Formula 3 gives

$$\begin{aligned} \int_C (2 + x^2y) ds &= \int_0^\pi (2 + \cos^2 t \sin t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\ &= \int_0^\pi (2 + \cos^2 t \sin t) \sqrt{\sin^2 t + \cos^2 t} dt \\ &= \int_0^\pi (2 + \cos^2 t \sin t) dt = \left[ 2t - \frac{\cos^3 t}{3} \right]_0^\pi \\ &= 2\pi + \frac{2}{3} \end{aligned}$$

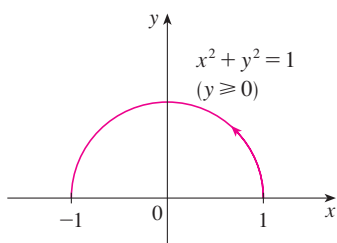


FIGURE 3

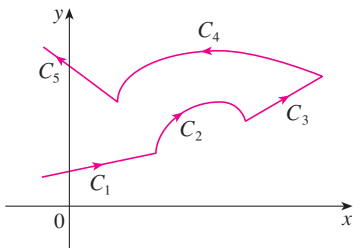


FIGURE 4

A piecewise-smooth curve

Suppose now that  $C$  is a **piecewise-smooth curve**; that is,  $C$  is a union of a finite number of smooth curves  $C_1, C_2, \dots, C_n$ , where, as illustrated in Figure 4, the initial point of  $C_{i+1}$  is the terminal point of  $C_i$ . Then we define the integral of  $f$  along  $C$  as the sum of the integrals of  $f$  along each of the smooth pieces of  $C$ :

$$\int_C f(x, y) ds = \int_{C_1} f(x, y) ds + \int_{C_2} f(x, y) ds + \cdots + \int_{C_n} f(x, y) ds$$

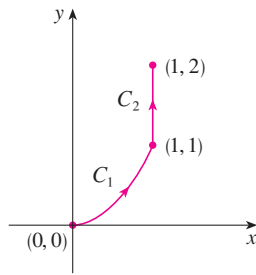


FIGURE 5  
 $C = C_1 \cup C_2$

**EXAMPLE 2** Evaluate  $\int_C 2x \, ds$ , where  $C$  consists of the arc  $C_1$  of the parabola  $y = x^2$  from  $(0, 0)$  to  $(1, 1)$  followed by the vertical line segment  $C_2$  from  $(1, 1)$  to  $(1, 2)$ .

**SOLUTION** The curve  $C$  is shown in Figure 5.  $C_1$  is the graph of a function of  $x$ , so we can choose  $x$  as the parameter and the equations for  $C_1$  become

$$x = x \quad y = x^2 \quad 0 \leq x \leq 1$$

Therefore

$$\begin{aligned} \int_{C_1} 2x \, ds &= \int_0^1 2x \sqrt{\left(\frac{dx}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2} \, dx = \int_0^1 2x \sqrt{1 + 4x^2} \, dx \\ &= \frac{1}{4} \cdot \frac{2}{3} (1 + 4x^2)^{3/2} \Big|_0^1 = \frac{5\sqrt{5} - 1}{6} \end{aligned}$$

On  $C_2$  we choose  $y$  as the parameter, so the equations of  $C_2$  are

$$x = 1 \quad y = y \quad 1 \leq y \leq 2$$

and

$$\int_{C_2} 2x \, ds = \int_1^2 2(1) \sqrt{\left(\frac{dx}{dy}\right)^2 + \left(\frac{dy}{dy}\right)^2} \, dy = \int_1^2 2 \, dy = 2$$

Thus

$$\int_C 2x \, ds = \int_{C_1} 2x \, ds + \int_{C_2} 2x \, ds = \frac{5\sqrt{5} - 1}{6} + 2$$

Any physical interpretation of a line integral  $\int_C f(x, y) \, ds$  depends on the physical interpretation of the function  $f$ . Suppose that  $\rho(x, y)$  represents the linear density at a point  $(x, y)$  of a thin wire shaped like a curve  $C$ . Then the mass of the part of the wire from  $P_{i-1}$  to  $P_i$  in Figure 1 is approximately  $\rho(x_i^*, y_i^*) \Delta s_i$  and so the total mass of the wire is approximately  $\sum \rho(x_i^*, y_i^*) \Delta s_i$ . By taking more and more points on the curve, we obtain the **mass**  $m$  of the wire as the limiting value of these approximations:

$$m = \lim_{n \rightarrow \infty} \sum_{i=1}^n \rho(x_i^*, y_i^*) \Delta s_i = \int_C \rho(x, y) \, ds$$

[For example, if  $f(x, y) = 2 + x^2y$  represents the density of a semicircular wire, then the integral in Example 1 would represent the mass of the wire.] The **center of mass** of the wire with density function  $\rho$  is located at the point  $(\bar{x}, \bar{y})$ , where

$$\boxed{4} \quad \bar{x} = \frac{1}{m} \int_C x \rho(x, y) \, ds \quad \bar{y} = \frac{1}{m} \int_C y \rho(x, y) \, ds$$

Other physical interpretations of line integrals will be discussed later in this chapter.

**EXAMPLE 3** A wire takes the shape of the semicircle  $x^2 + y^2 = 1$ ,  $y \geq 0$ , and is thicker near its base than near the top. Find the center of mass of the wire if the linear density at any point is proportional to its distance from the line  $y = 1$ .

**SOLUTION** As in Example 1 we use the parametrization  $x = \cos t$ ,  $y = \sin t$ ,  $0 \leq t \leq \pi$ , and find that  $ds = dt$ . The linear density is

$$\rho(x, y) = k(1 - y)$$

where  $k$  is a constant, and so the mass of the wire is

$$m = \int_C k(1 - y) ds = \int_0^\pi k(1 - \sin t) dt = k[t + \cos t]_0^\pi = k(\pi - 2)$$

From Equations 4 we have

$$\begin{aligned} \bar{y} &= \frac{1}{m} \int_C y \rho(x, y) ds = \frac{1}{k(\pi - 2)} \int_C y k(1 - y) ds \\ &= \frac{1}{\pi - 2} \int_0^\pi (\sin t - \sin^2 t) dt = \frac{1}{\pi - 2} \left[ -\cos t - \frac{1}{2}t + \frac{1}{4} \sin 2t \right]_0^\pi \\ &= \frac{4 - \pi}{2(\pi - 2)} \end{aligned}$$

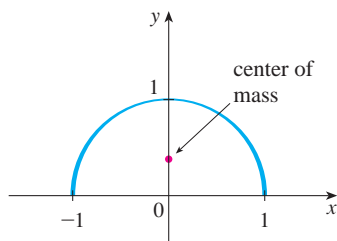


FIGURE 6

By symmetry we see that  $\bar{x} = 0$ , so the center of mass is

$$\left( 0, \frac{4 - \pi}{2(\pi - 2)} \right) \approx (0, 0.38)$$

See Figure 6.

Two other line integrals are obtained by replacing  $\Delta s_i$  by either  $\Delta x_i = x_i - x_{i-1}$  or  $\Delta y_i = y_i - y_{i-1}$  in Definition 2. They are called the **line integrals of  $f$  along  $C$  with respect to  $x$  and  $y$** :

$$\boxed{5} \quad \int_C f(x, y) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*, y_i^*) \Delta x_i$$

$$\boxed{6} \quad \int_C f(x, y) dy = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*, y_i^*) \Delta y_i$$

When we want to distinguish the original line integral  $\int_C f(x, y) ds$  from those in Equations 5 and 6, we call it the **line integral with respect to arc length**.

The following formulas say that line integrals with respect to  $x$  and  $y$  can also be evaluated by expressing everything in terms of  $t$ :  $x = x(t)$ ,  $y = y(t)$ ,  $dx = x'(t) dt$ ,  $dy = y'(t) dt$ .

$$\boxed{7} \quad \int_C f(x, y) dx = \int_a^b f(x(t), y(t)) x'(t) dt$$

$$\int_C f(x, y) dy = \int_a^b f(x(t), y(t)) y'(t) dt$$

It frequently happens that line integrals with respect to  $x$  and  $y$  occur together. When this happens, it's customary to abbreviate by writing

$$\int_C P(x, y) dx + \int_C Q(x, y) dy = \int_C P(x, y) dx + Q(x, y) dy$$

When we are setting up a line integral, sometimes the most difficult thing is to think of a parametric representation for a curve whose geometric description is given. In particular, we often need to parametrize a line segment, so it's useful to remember that a vector rep-

representation of the line segment that starts at  $\mathbf{r}_0$  and ends at  $\mathbf{r}_1$  is given by

$$\mathbf{r}(t) = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1 \quad 0 \leq t \leq 1$$

(See Equation 12.5.4.)

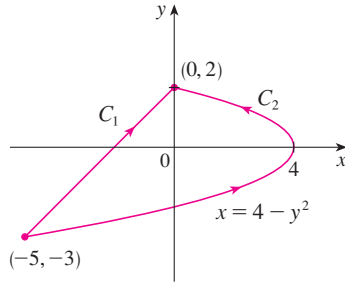


FIGURE 7

**V EXAMPLE 4** Evaluate  $\int_C y^2 dx + x dy$ , where (a)  $C = C_1$  is the line segment from  $(-5, -3)$  to  $(0, 2)$  and (b)  $C = C_2$  is the arc of the parabola  $x = 4 - y^2$  from  $(-5, -3)$  to  $(0, 2)$ . (See Figure 7.)

**SOLUTION**

(a) A parametric representation for the line segment is

$$x = 5t - 5 \quad y = 5t - 3 \quad 0 \leq t \leq 1$$

(Use Equation 8 with  $\mathbf{r}_0 = \langle -5, -3 \rangle$  and  $\mathbf{r}_1 = \langle 0, 2 \rangle$ .) Then  $dx = 5 dt$ ,  $dy = 5 dt$ , and Formulas 7 give

$$\begin{aligned} \int_{C_1} y^2 dx + x dy &= \int_0^1 (5t - 3)^2(5 dt) + (5t - 5)(5 dt) \\ &= 5 \int_0^1 (25t^2 - 25t + 4) dt \\ &= 5 \left[ \frac{25t^3}{3} - \frac{25t^2}{2} + 4t \right]_0^1 = -\frac{5}{6} \end{aligned}$$

(b) Since the parabola is given as a function of  $y$ , let's take  $y$  as the parameter and write  $C_2$  as

$$x = 4 - y^2 \quad y = y \quad -3 \leq y \leq 2$$

Then  $dx = -2y dy$  and by Formulas 7 we have

$$\begin{aligned} \int_{C_2} y^2 dx + x dy &= \int_{-3}^2 y^2(-2y) dy + (4 - y^2) dy \\ &= \int_{-3}^2 (-2y^3 - y^2 + 4) dy \\ &= \left[ -\frac{y^4}{2} - \frac{y^3}{3} + 4y \right]_{-3}^2 = 40\frac{5}{6} \end{aligned}$$

Notice that we got different answers in parts (a) and (b) of Example 4 even though the two curves had the same endpoints. Thus, in general, the value of a line integral depends not just on the endpoints of the curve but also on the path. (But see Section 16.3 for conditions under which the integral is independent of the path.)

Notice also that the answers in Example 4 depend on the direction, or orientation, of the curve. If  $-C_1$  denotes the line segment from  $(0, 2)$  to  $(-5, -3)$ , you can verify, using the parametrization

$$x = -5t \quad y = 2 - 5t \quad 0 \leq t \leq 1$$

that

$$\int_{-C_1} y^2 dx + x dy = \frac{5}{6}$$

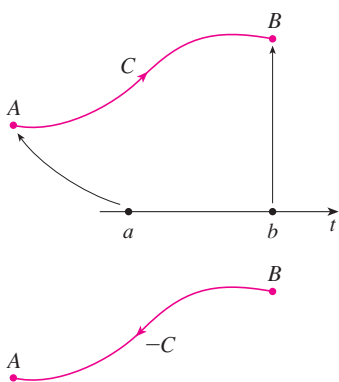


FIGURE 8

In general, a given parametrization  $x = x(t)$ ,  $y = y(t)$ ,  $a \leq t \leq b$ , determines an **orientation** of a curve  $C$ , with the positive direction corresponding to increasing values of the parameter  $t$ . (See Figure 8, where the initial point  $A$  corresponds to the parameter value  $a$  and the terminal point  $B$  corresponds to  $t = b$ .)

If  $-C$  denotes the curve consisting of the same points as  $C$  but with the opposite orientation (from initial point  $B$  to terminal point  $A$  in Figure 8), then we have

$$\int_{-C} f(x, y) dx = -\int_C f(x, y) dx \quad \int_{-C} f(x, y) dy = -\int_C f(x, y) dy$$

But if we integrate with respect to arc length, the value of the line integral does *not* change when we reverse the orientation of the curve:

$$\int_{-C} f(x, y) ds = \int_C f(x, y) ds$$

This is because  $\Delta s_i$  is always positive, whereas  $\Delta x_i$  and  $\Delta y_i$  change sign when we reverse the orientation of  $C$ .

### Line Integrals in Space

We now suppose that  $C$  is a smooth space curve given by the parametric equations

$$x = x(t) \quad y = y(t) \quad z = z(t) \quad a \leq t \leq b$$

or by a vector equation  $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ . If  $f$  is a function of three variables that is continuous on some region containing  $C$ , then we define the **line integral of  $f$  along  $C$**  (with respect to arc length) in a manner similar to that for plane curves:

$$\int_C f(x, y, z) ds = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*, y_i^*, z_i^*) \Delta s_i$$

We evaluate it using a formula similar to Formula 3:

$$\boxed{9} \quad \int_C f(x, y, z) ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt$$

Observe that the integrals in both Formulas 3 and 9 can be written in the more compact vector notation

$$\int_a^b f(\mathbf{r}(t)) |\mathbf{r}'(t)| dt$$

For the special case  $f(x, y, z) = 1$ , we get

$$\int_C ds = \int_a^b |\mathbf{r}'(t)| dt = L$$

where  $L$  is the length of the curve  $C$  (see Formula 13.3.3).

Line integrals along  $C$  with respect to  $x$ ,  $y$ , and  $z$  can also be defined. For example,

$$\begin{aligned}\int_C f(x, y, z) dz &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*, y_i^*, z_i^*) \Delta z_i \\ &= \int_a^b f(x(t), y(t), z(t)) z'(t) dt\end{aligned}$$

Therefore, as with line integrals in the plane, we evaluate integrals of the form

$$\int_C P(x, y, z) dx + Q(x, y, z) dy + R(x, y, z) dz$$

by expressing everything ( $x, y, z, dx, dy, dz$ ) in terms of the parameter  $t$ .

**V EXAMPLE 5** Evaluate  $\int_C y \sin z ds$ , where  $C$  is the circular helix given by the equations  $x = \cos t$ ,  $y = \sin t$ ,  $z = t$ ,  $0 \leq t \leq 2\pi$ . (See Figure 9.)

**SOLUTION** Formula 9 gives

$$\begin{aligned}\int_C y \sin z ds &= \int_0^{2\pi} (\sin t) \sin t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt \\ &= \int_0^{2\pi} \sin^2 t \sqrt{\sin^2 t + \cos^2 t + 1} dt = \sqrt{2} \int_0^{2\pi} \frac{1}{2}(1 - \cos 2t) dt \\ &= \frac{\sqrt{2}}{2} \left[ t - \frac{1}{2} \sin 2t \right]_0^{2\pi} = \sqrt{2} \pi\end{aligned}$$

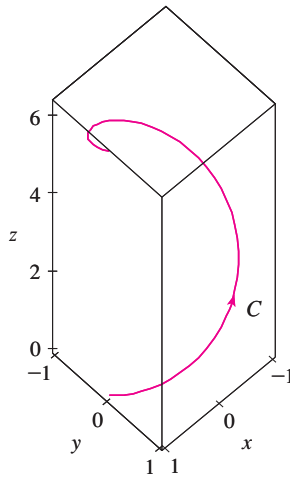


FIGURE 9

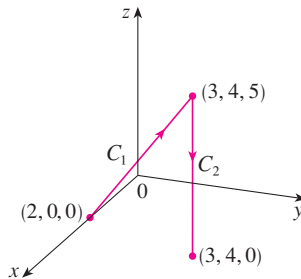


FIGURE 10

**EXAMPLE 6** Evaluate  $\int_C y dx + z dy + x dz$ , where  $C$  consists of the line segment  $C_1$  from  $(2, 0, 0)$  to  $(3, 4, 5)$ , followed by the vertical line segment  $C_2$  from  $(3, 4, 5)$  to  $(3, 4, 0)$ .

**SOLUTION** The curve  $C$  is shown in Figure 10. Using Equation 8, we write  $C_1$  as

$$\mathbf{r}(t) = (1 - t)\langle 2, 0, 0 \rangle + t\langle 3, 4, 5 \rangle = \langle 2 + t, 4t, 5t \rangle$$

or, in parametric form, as

$$x = 2 + t \quad y = 4t \quad z = 5t \quad 0 \leq t \leq 1$$

Thus

$$\begin{aligned}\int_{C_1} y dx + z dy + x dz &= \int_0^1 (4t) dt + (5t)4 dt + (2 + t)5 dt \\ &= \int_0^1 (10 + 29t) dt = 10t + 29 \frac{t^2}{2} \Big|_0^1 = 24.5\end{aligned}$$

Likewise,  $C_2$  can be written in the form

$$\mathbf{r}(t) = (1 - t)\langle 3, 4, 5 \rangle + t\langle 3, 4, 0 \rangle = \langle 3, 4, 5 - 5t \rangle$$

$$\text{or} \quad x = 3 \quad y = 4 \quad z = 5 - 5t \quad 0 \leq t \leq 1$$

Then  $dx = 0 = dy$ , so

$$\int_{C_2} y \, dx + z \, dy + x \, dz = \int_0^1 3(-5) \, dt = -15$$

Adding the values of these integrals, we obtain

$$\int_C y \, dx + z \, dy + x \, dz = 24.5 - 15 = 9.5$$

### Line Integrals of Vector Fields

Recall from Section 6.4 that the work done by a variable force  $f(x)$  in moving a particle from  $a$  to  $b$  along the  $x$ -axis is  $W = \int_a^b f(x) \, dx$ . Then in Section 12.3 we found that the work done by a constant force  $\mathbf{F}$  in moving an object from a point  $P$  to another point  $Q$  in space is  $W = \mathbf{F} \cdot \mathbf{D}$ , where  $\mathbf{D} = \overrightarrow{PQ}$  is the displacement vector.

Now suppose that  $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$  is a continuous force field on  $\mathbb{R}^3$ , such as the gravitational field of Example 4 in Section 16.1 or the electric force field of Example 5 in Section 16.1. (A force field on  $\mathbb{R}^2$  could be regarded as a special case where  $R = 0$  and  $P$  and  $Q$  depend only on  $x$  and  $y$ .) We wish to compute the work done by this force in moving a particle along a smooth curve  $C$ .

We divide  $C$  into subarcs  $P_{i-1}P_i$  with lengths  $\Delta s_i$  by dividing the parameter interval  $[a, b]$  into subintervals of equal width. (See Figure 1 for the two-dimensional case or Figure 11 for the three-dimensional case.) Choose a point  $P_i^*(x_i^*, y_i^*, z_i^*)$  on the  $i$ th subarc corresponding to the parameter value  $t_i^*$ . If  $\Delta s_i$  is small, then as the particle moves from  $P_{i-1}$  to  $P_i$  along the curve, it proceeds approximately in the direction of  $\mathbf{T}(t_i^*)$ , the unit tangent vector at  $P_i^*$ . Thus the work done by the force  $\mathbf{F}$  in moving the particle from  $P_{i-1}$  to  $P_i$  is approximately

$$\mathbf{F}(x_i^*, y_i^*, z_i^*) \cdot [\Delta s_i \mathbf{T}(t_i^*)] = [\mathbf{F}(x_i^*, y_i^*, z_i^*) \cdot \mathbf{T}(t_i^*)] \Delta s_i$$

and the total work done in moving the particle along  $C$  is approximately

$$\text{11} \quad \sum_{i=1}^n [\mathbf{F}(x_i^*, y_i^*, z_i^*) \cdot \mathbf{T}(x_i^*, y_i^*, z_i^*)] \Delta s_i$$

where  $\mathbf{T}(x, y, z)$  is the unit tangent vector at the point  $(x, y, z)$  on  $C$ . Intuitively, we see that these approximations ought to become better as  $n$  becomes larger. Therefore we define the **work**  $W$  done by the force field  $\mathbf{F}$  as the limit of the Riemann sums in [11](#), namely,

$$\text{12} \quad W = \int_C \mathbf{F}(x, y, z) \cdot \mathbf{T}(x, y, z) \, ds = \int_C \mathbf{F} \cdot \mathbf{T} \, ds$$

Equation 12 says that *work is the line integral with respect to arc length of the tangential component of the force*.

If the curve  $C$  is given by the vector equation  $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ , then  $\mathbf{T}(t) = \mathbf{r}'(t)/|\mathbf{r}'(t)|$ , so using Equation 9 we can rewrite Equation 12 in the form

$$W = \int_a^b \left[ \mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \right] |\mathbf{r}'(t)| \, dt = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt$$

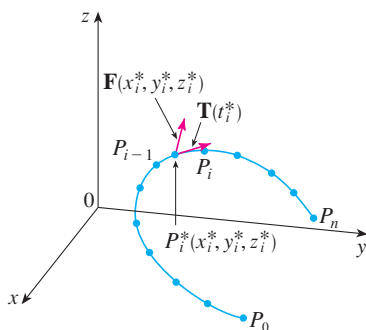


FIGURE 11



This integral is often abbreviated as  $\int_C \mathbf{F} \cdot d\mathbf{r}$  and occurs in other areas of physics as well. Therefore we make the following definition for the line integral of *any* continuous vector field.

**13 Definition** Let  $\mathbf{F}$  be a continuous vector field defined on a smooth curve  $C$  given by a vector function  $\mathbf{r}(t)$ ,  $a \leq t \leq b$ . Then the **line integral of  $\mathbf{F}$  along  $C$**  is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt = \int_C \mathbf{F} \cdot \mathbf{T} ds$$

When using Definition 13, bear in mind that  $\mathbf{F}(\mathbf{r}(t))$  is just an abbreviation for  $\mathbf{F}(x(t), y(t), z(t))$ , so we evaluate  $\mathbf{F}(\mathbf{r}(t))$  simply by putting  $x = x(t)$ ,  $y = y(t)$ , and  $z = z(t)$  in the expression for  $\mathbf{F}(x, y, z)$ . Notice also that we can formally write  $d\mathbf{r} = \mathbf{r}'(t) dt$ .

Figure 12 shows the force field and the curve in Example 7. The work done is negative because the field impedes movement along the curve.

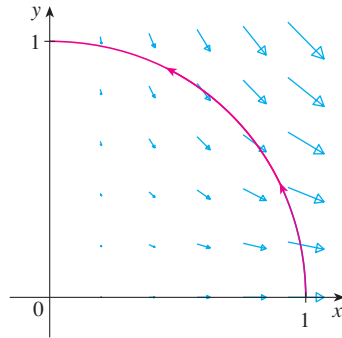


FIGURE 12

**EXAMPLE 7** Find the work done by the force field  $\mathbf{F}(x, y) = x^2 \mathbf{i} - xy \mathbf{j}$  in moving a particle along the quarter-circle  $\mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j}$ ,  $0 \leq t \leq \pi/2$ .

**SOLUTION** Since  $x = \cos t$  and  $y = \sin t$ , we have

$$\mathbf{F}(\mathbf{r}(t)) = \cos^2 t \mathbf{i} - \cos t \sin t \mathbf{j}$$

and

$$\mathbf{r}'(t) = -\sin t \mathbf{i} + \cos t \mathbf{j}$$

Therefore the work done is

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{\pi/2} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt = \int_0^{\pi/2} (-2 \cos^2 t \sin t) dt \\ &= 2 \left. \frac{\cos^3 t}{3} \right|_0^{\pi/2} = -\frac{2}{3} \end{aligned}$$

**NOTE** Even though  $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T} ds$  and integrals with respect to arc length are unchanged when orientation is reversed, it is still true that

$$\int_{-C} \mathbf{F} \cdot d\mathbf{r} = -\int_C \mathbf{F} \cdot d\mathbf{r}$$

because the unit tangent vector  $\mathbf{T}$  is replaced by its negative when  $C$  is replaced by  $-C$ .

**EXAMPLE 8** Evaluate  $\int_C \mathbf{F} \cdot d\mathbf{r}$ , where  $\mathbf{F}(x, y, z) = xy \mathbf{i} + yz \mathbf{j} + zx \mathbf{k}$  and  $C$  is the twisted cubic given by

$$x = t \quad y = t^2 \quad z = t^3 \quad 0 \leq t \leq 1$$

**SOLUTION** We have

$$\mathbf{r}(t) = t \mathbf{i} + t^2 \mathbf{j} + t^3 \mathbf{k}$$

$$\mathbf{r}'(t) = \mathbf{i} + 2t \mathbf{j} + 3t^2 \mathbf{k}$$

$$\mathbf{F}(\mathbf{r}(t)) = t^3 \mathbf{i} + t^5 \mathbf{j} + t^4 \mathbf{k}$$

Figure 13 shows the twisted cubic  $C$  in Example 8 and some typical vectors acting at three points on  $C$ .

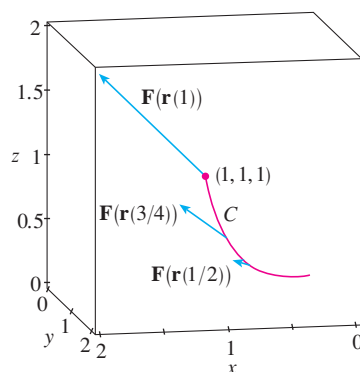


FIGURE 13

$$\begin{aligned}\text{Thus} \quad \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt \\ &= \int_0^1 (t^3 + 5t^6) dt = \left. \frac{t^4}{4} + \frac{5t^7}{7} \right|_0^1 = \frac{27}{28}\end{aligned}$$

Finally, we note the connection between line integrals of vector fields and line integrals of scalar fields. Suppose the vector field  $\mathbf{F}$  on  $\mathbb{R}^3$  is given in component form by the equation  $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ . We use Definition 13 to compute its line integral along  $C$ :

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt \\ &= \int_a^b (P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}) \cdot (x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}) dt \\ &= \int_a^b [P(x(t), y(t), z(t))x'(t) + Q(x(t), y(t), z(t))y'(t) + R(x(t), y(t), z(t))z'(t)] dt\end{aligned}$$

But this last integral is precisely the line integral in [10]. Therefore we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P dx + Q dy + R dz \quad \text{where } \mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$$

For example, the integral  $\int_C y dx + z dy + x dz$  in Example 6 could be expressed as  $\int_C \mathbf{F} \cdot d\mathbf{r}$  where

$$\mathbf{F}(x, y, z) = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$$

## 16.2 Exercises

**1–16** Evaluate the line integral, where  $C$  is the given curve.

1.  $\int_C y^3 ds$ ,  $C: x = t^3, y = t, 0 \leq t \leq 2$

2.  $\int_C xy ds$ ,  $C: x = t^2, y = 2t, 0 \leq t \leq 1$

3.  $\int_C xy^4 ds$ ,  $C$  is the right half of the circle  $x^2 + y^2 = 16$

4.  $\int_C x \sin y ds$ ,  $C$  is the line segment from  $(0, 3)$  to  $(4, 6)$

5.  $\int_C (x^2y^3 - \sqrt{x}) dy$ ,  
 $C$  is the arc of the curve  $y = \sqrt{x}$  from  $(1, 1)$  to  $(4, 2)$

6.  $\int_C e^x dx$ ,  
 $C$  is the arc of the curve  $x = y^3$  from  $(-1, -1)$  to  $(1, 1)$

7.  $\int_C (x + 2y) dx + x^2 dy$ ,  $C$  consists of line segments from  $(0, 0)$  to  $(2, 1)$  and from  $(2, 1)$  to  $(3, 0)$

8.  $\int_C x^2 dx + y^2 dy$ ,  $C$  consists of the arc of the circle  $x^2 + y^2 = 4$  from  $(2, 0)$  to  $(0, 2)$  followed by the line segment from  $(0, 2)$  to  $(4, 3)$

9.  $\int_C xyz ds$ ,  
 $C: x = 2 \sin t, y = t, z = -2 \cos t, 0 \leq t \leq \pi$

10.  $\int_C xyz^2 ds$ ,  
 $C$  is the line segment from  $(-1, 5, 0)$  to  $(1, 6, 4)$

11.  $\int_C xe^{yz} ds$ ,  
 $C$  is the line segment from  $(0, 0, 0)$  to  $(1, 2, 3)$

12.  $\int_C (x^2 + y^2 + z^2) ds$ ,  
 $C: x = t, y = \cos 2t, z = \sin 2t, 0 \leq t \leq 2\pi$

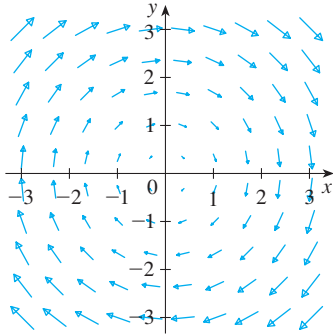
13.  $\int_C xye^{yz} dy$ ,  $C: x = t, y = t^2, z = t^3, 0 \leq t \leq 1$

14.  $\int_C y dx + z dy + x dz$ ,  
 $C: x = \sqrt{t}, y = t, z = t^2, 1 \leq t \leq 4$

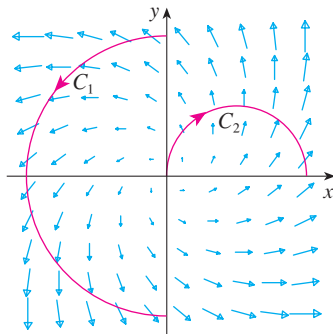
15.  $\int_C z^2 dx + x^2 dy + y^2 dz$ ,  $C$  is the line segment from  $(1, 0, 0)$  to  $(4, 1, 2)$

16.  $\int_C (y + z) dx + (x + z) dy + (x + y) dz$ ,  $C$  consists of line segments from  $(0, 0, 0)$  to  $(1, 0, 1)$  and from  $(1, 0, 1)$  to  $(0, 1, 2)$

17. Let  $\mathbf{F}$  be the vector field shown in the figure.
- (a) If  $C_1$  is the vertical line segment from  $(-3, -3)$  to  $(-3, 3)$ , determine whether  $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$  is positive, negative, or zero.
- (b) If  $C_2$  is the counterclockwise-oriented circle with radius 3 and center the origin, determine whether  $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$  is positive, negative, or zero.



18. The figure shows a vector field  $\mathbf{F}$  and two curves  $C_1$  and  $C_2$ . Are the line integrals of  $\mathbf{F}$  over  $C_1$  and  $C_2$  positive, negative, or zero? Explain.



- 19–22 Evaluate the line integral  $\int_C \mathbf{F} \cdot d\mathbf{r}$ , where  $C$  is given by the vector function  $\mathbf{r}(t)$ .

19.  $\mathbf{F}(x, y) = xy\mathbf{i} + 3y^2\mathbf{j}$ ,  
 $\mathbf{r}(t) = 11t^4\mathbf{i} + t^3\mathbf{j}$ ,  $0 \leq t \leq 1$
20.  $\mathbf{F}(x, y, z) = (x + y)\mathbf{i} + (y - z)\mathbf{j} + z^2\mathbf{k}$ ,  
 $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j} + t^2\mathbf{k}$ ,  $0 \leq t \leq 1$
21.  $\mathbf{F}(x, y, z) = \sin x\mathbf{i} + \cos y\mathbf{j} + xz\mathbf{k}$ ,  
 $\mathbf{r}(t) = t^3\mathbf{i} - t^2\mathbf{j} + t\mathbf{k}$ ,  $0 \leq t \leq 1$
22.  $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + xy\mathbf{k}$ ,  
 $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$ ,  $0 \leq t \leq \pi$

- 23–26 Use a calculator or CAS to evaluate the line integral correct to four decimal places.

23.  $\int_C \mathbf{F} \cdot d\mathbf{r}$ , where  $\mathbf{F}(x, y) = xy\mathbf{i} + \sin y\mathbf{j}$  and  
 $\mathbf{r}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j}$ ,  $1 \leq t \leq 2$


24.  $\int_C \mathbf{F} \cdot d\mathbf{r}$ , where  $\mathbf{F}(x, y, z) = y \sin z\mathbf{i} + z \sin x\mathbf{j} + x \sin y\mathbf{k}$  and  $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + \sin 5t\mathbf{k}$ ,  $0 \leq t \leq \pi$
25.  $\int_C x \sin(y + z) ds$ , where  $C$  has parametric equations  $x = t^2$ ,  $y = t^3$ ,  $z = t^4$ ,  $0 \leq t \leq 5$
26.  $\int_C ze^{-xy} ds$ , where  $C$  has parametric equations  $x = t$ ,  $y = t^2$ ,  $z = e^{-t}$ ,  $0 \leq t \leq 1$

- CAS** 27–28 Use a graph of the vector field  $\mathbf{F}$  and the curve  $C$  to guess whether the line integral of  $\mathbf{F}$  over  $C$  is positive, negative, or zero. Then evaluate the line integral.


27.  $\mathbf{F}(x, y) = (x - y)\mathbf{i} + xy\mathbf{j}$ ,  
 $C$  is the arc of the circle  $x^2 + y^2 = 4$  traversed counterclockwise from  $(2, 0)$  to  $(0, -2)$

28.  $\mathbf{F}(x, y) = \frac{x}{\sqrt{x^2 + y^2}}\mathbf{i} + \frac{y}{\sqrt{x^2 + y^2}}\mathbf{j}$ ,  
 $C$  is the parabola  $y = 1 + x^2$  from  $(-1, 2)$  to  $(1, 2)$

29. (a) Evaluate the line integral  $\int_C \mathbf{F} \cdot d\mathbf{r}$ , where  
 $\mathbf{F}(x, y) = e^{x-1}\mathbf{i} + xy\mathbf{j}$  and  $C$  is given by  
 $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j}$ ,  $0 \leq t \leq 1$ .

-  (b) Illustrate part (a) by using a graphing calculator or computer to graph  $C$  and the vectors from the vector field corresponding to  $t = 0, 1/\sqrt{2}$ , and 1 (as in Figure 13).

30. (a) Evaluate the line integral  $\int_C \mathbf{F} \cdot d\mathbf{r}$ , where  
 $\mathbf{F}(x, y, z) = x\mathbf{i} - z\mathbf{j} + y\mathbf{k}$  and  $C$  is given by  
 $\mathbf{r}(t) = 2t\mathbf{i} + 3t\mathbf{j} - t^2\mathbf{k}$ ,  $-1 \leq t \leq 1$ .

-  (b) Illustrate part (a) by using a computer to graph  $C$  and the vectors from the vector field corresponding to  $t = \pm 1$  and  $\pm \frac{1}{2}$  (as in Figure 13).

- CAS** 31. Find the exact value of  $\int_C x^3 y^2 z ds$ , where  $C$  is the curve with parametric equations  $x = e^{-t} \cos 4t$ ,  $y = e^{-t} \sin 4t$ ,  $z = e^{-t}$ ,  $0 \leq t \leq 2\pi$ .

32. (a) Find the work done by the force field  $\mathbf{F}(x, y) = x^2\mathbf{i} + xy\mathbf{j}$  on a particle that moves once around the circle  $x^2 + y^2 = 4$  oriented in the counter-clockwise direction.

- CAS** (b) Use a computer algebra system to graph the force field and circle on the same screen. Use the graph to explain your answer to part (a).

33. A thin wire is bent into the shape of a semicircle  $x^2 + y^2 = 4$ ,  $x \geq 0$ . If the linear density is a constant  $k$ , find the mass and center of mass of the wire.

34. A thin wire has the shape of the first-quadrant part of the circle with center the origin and radius  $a$ . If the density function is  $\rho(x, y) = kxy$ , find the mass and center of mass of the wire.

35. (a) Write the formulas similar to Equations 4 for the center of mass  $(\bar{x}, \bar{y}, \bar{z})$  of a thin wire in the shape of a space curve  $C$  if the wire has density function  $\rho(x, y, z)$ .

- (b) Find the center of mass of a wire in the shape of the helix  $x = 2 \sin t$ ,  $y = 2 \cos t$ ,  $z = 3t$ ,  $0 \leq t \leq 2\pi$ , if the density is a constant  $k$ .
36. Find the mass and center of mass of a wire in the shape of the helix  $x = t$ ,  $y = \cos t$ ,  $z = \sin t$ ,  $0 \leq t \leq 2\pi$ , if the density at any point is equal to the square of the distance from the origin.
37. If a wire with linear density  $\rho(x, y)$  lies along a plane curve  $C$ , its **moments of inertia** about the  $x$ - and  $y$ -axes are defined as

$$I_x = \int_C y^2 \rho(x, y) \, ds \quad I_y = \int_C x^2 \rho(x, y) \, ds$$

Find the moments of inertia for the wire in Example 3.

38. If a wire with linear density  $\rho(x, y, z)$  lies along a space curve  $C$ , its **moments of inertia** about the  $x$ -,  $y$ -, and  $z$ -axes are defined as

$$I_x = \int_C (y^2 + z^2) \rho(x, y, z) \, ds$$

$$I_y = \int_C (x^2 + z^2) \rho(x, y, z) \, ds$$

$$I_z = \int_C (x^2 + y^2) \rho(x, y, z) \, ds$$

Find the moments of inertia for the wire in Exercise 35.

39. Find the work done by the force field  $\mathbf{F}(x, y) = x \mathbf{i} + (y + 2) \mathbf{j}$  in moving an object along an arch of the cycloid  $\mathbf{r}(t) = (t - \sin t) \mathbf{i} + (1 - \cos t) \mathbf{j}$ ,  $0 \leq t \leq 2\pi$ .
40. Find the work done by the force field  $\mathbf{F}(x, y) = x^2 \mathbf{i} + ye^x \mathbf{j}$  on a particle that moves along the parabola  $x = y^2 + 1$  from  $(1, 0)$  to  $(2, 1)$ .
41. Find the work done by the force field  $\mathbf{F}(x, y, z) = \langle x - y^2, y - z^2, z - x^2 \rangle$  on a particle that moves along the line segment from  $(0, 0, 1)$  to  $(2, 1, 0)$ .
42. The force exerted by an electric charge at the origin on a charged particle at a point  $(x, y, z)$  with position vector  $\mathbf{r} = \langle x, y, z \rangle$  is  $\mathbf{F}(\mathbf{r}) = K\mathbf{r}/|\mathbf{r}|^3$  where  $K$  is a constant. (See Example 5 in Section 16.1.) Find the work done as the particle moves along a straight line from  $(2, 0, 0)$  to  $(2, 1, 5)$ .
43. The position of an object with mass  $m$  at time  $t$  is  $\mathbf{r}(t) = at^2 \mathbf{i} + bt^3 \mathbf{j}$ ,  $0 \leq t \leq 1$ .
- (a) What is the force acting on the object at time  $t$ ?
- (b) What is the work done by the force during the time interval  $0 \leq t \leq 1$ ?
44. An object with mass  $m$  moves with position function  $\mathbf{r}(t) = a \sin t \mathbf{i} + b \cos t \mathbf{j} + ct \mathbf{k}$ ,  $0 \leq t \leq \pi/2$ . Find the work done on the object during this time period.

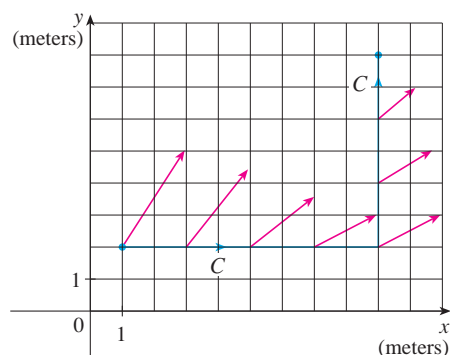
45. A 160-lb man carries a 25-lb can of paint up a helical staircase that encircles a silo with a radius of 20 ft. If the silo is 90 ft high and the man makes exactly three complete revolutions climbing to the top, how much work is done by the man against gravity?
46. Suppose there is a hole in the can of paint in Exercise 45 and 9 lb of paint leaks steadily out of the can during the man's ascent. How much work is done?
47. (a) Show that a constant force field does zero work on a particle that moves once uniformly around the circle  $x^2 + y^2 = 1$ .
- (b) Is this also true for a force field  $\mathbf{F}(\mathbf{x}) = k\mathbf{x}$ , where  $k$  is a constant and  $\mathbf{x} = \langle x, y \rangle$ ?
48. The base of a circular fence with radius 10 m is given by  $x = 10 \cos t$ ,  $y = 10 \sin t$ . The height of the fence at position  $(x, y)$  is given by the function  $h(x, y) = 4 + 0.01(x^2 - y^2)$ , so the height varies from 3 m to 5 m. Suppose that 1 L of paint covers  $100 \text{ m}^2$ . Sketch the fence and determine how much paint you will need if you paint both sides of the fence.
49. If  $C$  is a smooth curve given by a vector function  $\mathbf{r}(t)$ ,  $a \leq t \leq b$ , and  $\mathbf{v}$  is a constant vector, show that

$$\int_C \mathbf{v} \cdot d\mathbf{r} = \mathbf{v} \cdot [\mathbf{r}(b) - \mathbf{r}(a)]$$

50. If  $C$  is a smooth curve given by a vector function  $\mathbf{r}(t)$ ,  $a \leq t \leq b$ , show that

$$\int_C \mathbf{r} \cdot d\mathbf{r} = \frac{1}{2} [|\mathbf{r}(b)|^2 - |\mathbf{r}(a)|^2]$$

51. An object moves along the curve  $C$  shown in the figure from  $(1, 2)$  to  $(9, 8)$ . The lengths of the vectors in the force field  $\mathbf{F}$  are measured in newtons by the scales on the axes. Estimate the work done by  $\mathbf{F}$  on the object.



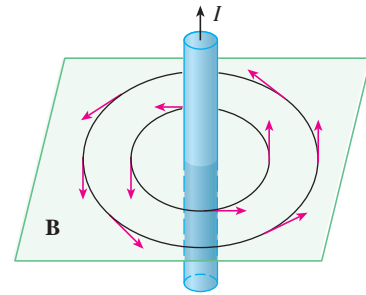
52. Experiments show that a steady current  $I$  in a long wire produces a magnetic field  $\mathbf{B}$  that is tangent to any circle that lies in the plane perpendicular to the wire and whose center is the axis of the wire (as in the figure). *Ampère's Law* relates the electric

current to its magnetic effects and states that

$$\int_C \mathbf{B} \cdot d\mathbf{r} = \mu_0 I$$

where  $I$  is the net current that passes through any surface bounded by a closed curve  $C$ , and  $\mu_0$  is a constant called the permeability of free space. By taking  $C$  to be a circle with radius  $r$ , show that the magnitude  $B = |\mathbf{B}|$  of the magnetic field at a distance  $r$  from the center of the wire is

$$B = \frac{\mu_0 I}{2\pi r}$$



## 16.3 The Fundamental Theorem for Line Integrals

Recall from Section 5.3 that Part 2 of the Fundamental Theorem of Calculus can be written as

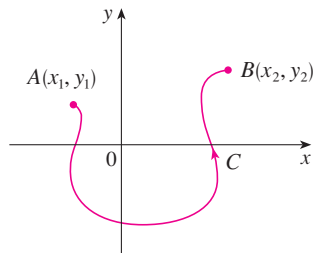
$$\boxed{1} \quad \int_a^b F'(x) dx = F(b) - F(a)$$

where  $F'$  is continuous on  $[a, b]$ . We also called Equation 1 the Net Change Theorem: The integral of a rate of change is the net change.

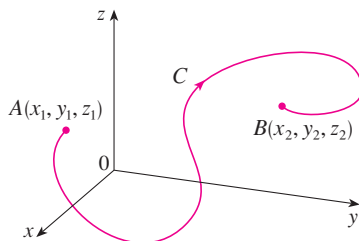
If we think of the gradient vector  $\nabla f$  of a function  $f$  of two or three variables as a sort of derivative of  $f$ , then the following theorem can be regarded as a version of the Fundamental Theorem for line integrals.

**2 Theorem** Let  $C$  be a smooth curve given by the vector function  $\mathbf{r}(t)$ ,  $a \leq t \leq b$ . Let  $f$  be a differentiable function of two or three variables whose gradient vector  $\nabla f$  is continuous on  $C$ . Then

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$



(a)



(b)

FIGURE 1

**NOTE** Theorem 2 says that we can evaluate the line integral of a conservative vector field (the gradient vector field of the potential function  $f$ ) simply by knowing the value of  $f$  at the endpoints of  $C$ . In fact, Theorem 2 says that the line integral of  $\nabla f$  is the net change in  $f$ . If  $f$  is a function of two variables and  $C$  is a plane curve with initial point  $A(x_1, y_1)$  and terminal point  $B(x_2, y_2)$ , as in Figure 1, then Theorem 2 becomes

$$\int_C \nabla f \cdot d\mathbf{r} = f(x_2, y_2) - f(x_1, y_1)$$

If  $f$  is a function of three variables and  $C$  is a space curve joining the point  $A(x_1, y_1, z_1)$  to the point  $B(x_2, y_2, z_2)$ , then we have

$$\int_C \nabla f \cdot d\mathbf{r} = f(x_2, y_2, z_2) - f(x_1, y_1, z_1)$$

Let's prove Theorem 2 for this case.

**PROOF OF THEOREM 2** Using Definition 16.2.13, we have

$$\begin{aligned}\int_C \nabla f \cdot d\mathbf{r} &= \int_a^b \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt \\ &= \int_a^b \left( \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt} \right) dt \\ &= \int_a^b \frac{d}{dt} f(\mathbf{r}(t)) dt \quad (\text{by the Chain Rule}) \\ &= f(\mathbf{r}(b)) - f(\mathbf{r}(a))\end{aligned}$$

The last step follows from the Fundamental Theorem of Calculus (Equation 1). ■

Although we have proved Theorem 2 for smooth curves, it is also true for piecewise-smooth curves. This can be seen by subdividing  $C$  into a finite number of smooth curves and adding the resulting integrals.

**EXAMPLE 1** Find the work done by the gravitational field

$$\mathbf{F}(\mathbf{x}) = -\frac{mMG}{|\mathbf{x}|^3} \mathbf{x}$$

in moving a particle with mass  $m$  from the point  $(3, 4, 12)$  to the point  $(2, 2, 0)$  along a piecewise-smooth curve  $C$ . (See Example 4 in Section 16.1.)

**SOLUTION** From Section 16.1 we know that  $\mathbf{F}$  is a conservative vector field and, in fact,  $\mathbf{F} = \nabla f$ , where

$$f(x, y, z) = \frac{mMG}{\sqrt{x^2 + y^2 + z^2}}$$

Therefore, by Theorem 2, the work done is

$$\begin{aligned}W &= \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} \\ &= f(2, 2, 0) - f(3, 4, 12) \\ &= \frac{mMG}{\sqrt{2^2 + 2^2}} - \frac{mMG}{\sqrt{3^2 + 4^2 + 12^2}} = mMG \left( \frac{1}{2\sqrt{2}} - \frac{1}{13} \right)\end{aligned}$$
■

### Independence of Path

Suppose  $C_1$  and  $C_2$  are two piecewise-smooth curves (which are called **paths**) that have the same initial point  $A$  and terminal point  $B$ . We know from Example 4 in Section 16.2 that, in general,  $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} \neq \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ . But one implication of Theorem 2 is that

$$\int_{C_1} \nabla f \cdot d\mathbf{r} = \int_{C_2} \nabla f \cdot d\mathbf{r}$$

whenever  $\nabla f$  is continuous. In other words, the line integral of a *conservative* vector field depends only on the initial point and terminal point of a curve.

In general, if  $\mathbf{F}$  is a continuous vector field with domain  $D$ , we say that the line integral  $\int_C \mathbf{F} \cdot d\mathbf{r}$  is **independent of path** if  $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$  for any two paths  $C_1$  and  $C_2$  in  $D$  that have the same initial and terminal points. With this terminology we can say that *line integrals of conservative vector fields are independent of path*.

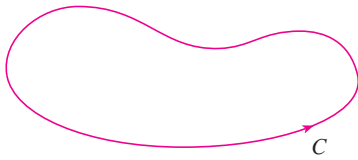


FIGURE 2

A closed curve

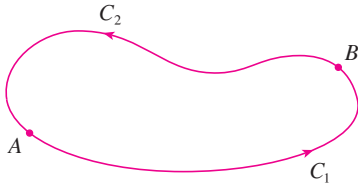


FIGURE 3

A curve is called **closed** if its terminal point coincides with its initial point, that is,  $\mathbf{r}(b) = \mathbf{r}(a)$ . (See Figure 2.) If  $\int_C \mathbf{F} \cdot d\mathbf{r}$  is independent of path in  $D$  and  $C$  is any closed path in  $D$ , we can choose any two points  $A$  and  $B$  on  $C$  and regard  $C$  as being composed of the path  $C_1$  from  $A$  to  $B$  followed by the path  $C_2$  from  $B$  to  $A$ . (See Figure 3.) Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{-C_2} \mathbf{F} \cdot d\mathbf{r} = 0$$

since  $C_1$  and  $-C_2$  have the same initial and terminal points.

Conversely, if it is true that  $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$  whenever  $C$  is a closed path in  $D$ , then we demonstrate independence of path as follows. Take any two paths  $C_1$  and  $C_2$  from  $A$  to  $B$  in  $D$  and define  $C$  to be the curve consisting of  $C_1$  followed by  $-C_2$ . Then

$$0 = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{-C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

and so  $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ . Thus we have proved the following theorem.

**3 Theorem**  $\int_C \mathbf{F} \cdot d\mathbf{r}$  is independent of path in  $D$  if and only if  $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$  for every closed path  $C$  in  $D$ .

Since we know that the line integral of any conservative vector field  $\mathbf{F}$  is independent of path, it follows that  $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$  for any closed path. The physical interpretation is that the work done by a conservative force field (such as the gravitational or electric field in Section 16.1) as it moves an object around a closed path is 0.

The following theorem says that the *only* vector fields that are independent of path are conservative. It is stated and proved for plane curves, but there is a similar version for space curves. We assume that  $D$  is **open**, which means that for every point  $P$  in  $D$  there is a disk with center  $P$  that lies entirely in  $D$ . (So  $D$  doesn't contain any of its boundary points.) In addition, we assume that  $D$  is **connected**: This means that any two points in  $D$  can be joined by a path that lies in  $D$ .

**4 Theorem** Suppose  $\mathbf{F}$  is a vector field that is continuous on an open connected region  $D$ . If  $\int_C \mathbf{F} \cdot d\mathbf{r}$  is independent of path in  $D$ , then  $\mathbf{F}$  is a conservative vector field on  $D$ ; that is, there exists a function  $f$  such that  $\nabla f = \mathbf{F}$ .

**PROOF** Let  $A(a, b)$  be a fixed point in  $D$ . We construct the desired potential function  $f$  by defining

$$f(x, y) = \int_{(a, b)}^{(x, y)} \mathbf{F} \cdot d\mathbf{r}$$

for any point  $(x, y)$  in  $D$ . Since  $\int_C \mathbf{F} \cdot d\mathbf{r}$  is independent of path, it does not matter which path  $C$  from  $(a, b)$  to  $(x, y)$  is used to evaluate  $f(x, y)$ . Since  $D$  is open, there exists a disk contained in  $D$  with center  $(x, y)$ . Choose any point  $(x_1, y)$  in the disk with  $x_1 < x$  and let  $C$  consist of any path  $C_1$  from  $(a, b)$  to  $(x_1, y)$  followed by the horizontal line segment  $C_2$  from  $(x_1, y)$  to  $(x, y)$ . (See Figure 4.) Then

$$f(x, y) = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{(a, b)}^{(x_1, y)} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

Notice that the first of these integrals does not depend on  $x$ , so

$$\frac{\partial}{\partial x} f(x, y) = 0 + \frac{\partial}{\partial x} \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

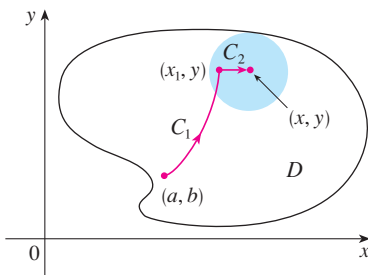


FIGURE 4

If we write  $\mathbf{F} = P \mathbf{i} + Q \mathbf{j}$ , then

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} P dx + Q dy$$

On  $C_2$ ,  $y$  is constant, so  $dy = 0$ . Using  $t$  as the parameter, where  $x_1 \leq t \leq x$ , we have

$$\frac{\partial}{\partial x} f(x, y) = \frac{\partial}{\partial x} \int_{C_2} P dx + Q dy = \frac{\partial}{\partial x} \int_{x_1}^x P(t, y) dt = P(x, y)$$

by Part 1 of the Fundamental Theorem of Calculus (see Section 5.3). A similar argument, using a vertical line segment (see Figure 5), shows that

$$\frac{\partial}{\partial y} f(x, y) = \frac{\partial}{\partial y} \int_{C_1} P dx + Q dy = \frac{\partial}{\partial y} \int_{y_1}^y Q(x, t) dt = Q(x, y)$$

Thus 
$$\mathbf{F} = P \mathbf{i} + Q \mathbf{j} = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} = \nabla f$$

which says that  $\mathbf{F}$  is conservative.

The question remains: How is it possible to determine whether or not a vector field  $\mathbf{F}$  is conservative? Suppose it is known that  $\mathbf{F} = P \mathbf{i} + Q \mathbf{j}$  is conservative, where  $P$  and  $Q$  have continuous first-order partial derivatives. Then there is a function  $f$  such that  $\mathbf{F} = \nabla f$ , that is,

$$P = \frac{\partial f}{\partial x} \quad \text{and} \quad Q = \frac{\partial f}{\partial y}$$

Therefore, by Clairaut's Theorem,

$$\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial Q}{\partial x}$$

**5 Theorem** If  $\mathbf{F}(x, y) = P(x, y) \mathbf{i} + Q(x, y) \mathbf{j}$  is a conservative vector field, where  $P$  and  $Q$  have continuous first-order partial derivatives on a domain  $D$ , then throughout  $D$  we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

The converse of Theorem 5 is true only for a special type of region. To explain this, we first need the concept of a **simple curve**, which is a curve that doesn't intersect itself anywhere between its endpoints. [See Figure 6;  $\mathbf{r}(a) = \mathbf{r}(b)$  for a simple closed curve, but  $\mathbf{r}(t_1) \neq \mathbf{r}(t_2)$  when  $a < t_1 < t_2 < b$ .]

In Theorem 4 we needed an open connected region. For the next theorem we need a stronger condition. A **simply-connected region** in the plane is a connected region  $D$  such that every simple closed curve in  $D$  encloses only points that are in  $D$ . Notice from Figure 7 that, intuitively speaking, a simply-connected region contains no hole and can't consist of two separate pieces.

In terms of simply-connected regions, we can now state a partial converse to Theorem 5 that gives a convenient method for verifying that a vector field on  $\mathbb{R}^2$  is conservative. The proof will be sketched in the next section as a consequence of Green's Theorem.

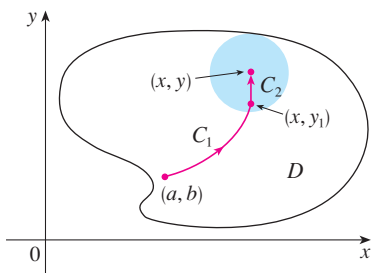


FIGURE 5

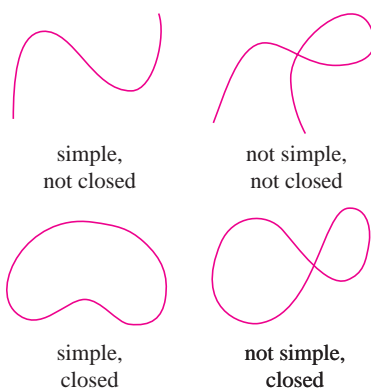


FIGURE 6  
Types of curves

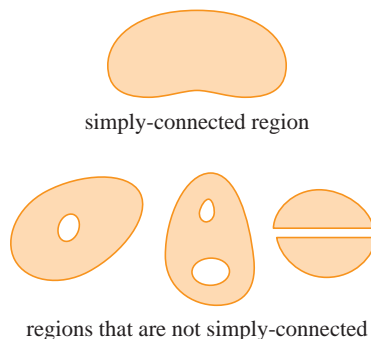


FIGURE 7



**6 Theorem** Let  $\mathbf{F} = P \mathbf{i} + Q \mathbf{j}$  be a vector field on an open simply-connected region  $D$ . Suppose that  $P$  and  $Q$  have continuous first-order derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \text{throughout } D$$

Then  $\mathbf{F}$  is conservative.

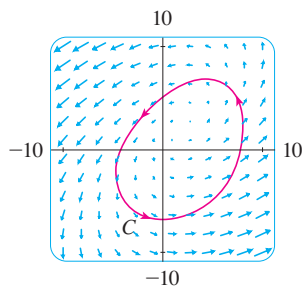


FIGURE 8

Figures 8 and 9 show the vector fields in Examples 2 and 3, respectively. The vectors in Figure 8 that start on the closed curve  $C$  all appear to point in roughly the same direction as  $C$ . So it looks as if  $\int_C \mathbf{F} \cdot d\mathbf{r} > 0$  and therefore  $\mathbf{F}$  is not conservative. The calculation in Example 2 confirms this impression. Some of the vectors near the curves  $C_1$  and  $C_2$  in Figure 9 point in approximately the same direction as the curves, whereas others point in the opposite direction. So it appears plausible that line integrals around all closed paths are 0. Example 3 shows that  $\mathbf{F}$  is indeed conservative.

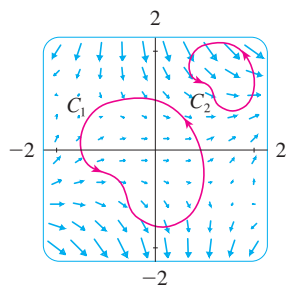


FIGURE 9

**V EXAMPLE 2** Determine whether or not the vector field

$$\mathbf{F}(x, y) = (x - y) \mathbf{i} + (x - 2) \mathbf{j}$$

is conservative.

**SOLUTION** Let  $P(x, y) = x - y$  and  $Q(x, y) = x - 2$ . Then

$$\frac{\partial P}{\partial y} = -1 \quad \frac{\partial Q}{\partial x} = 1$$

Since  $\partial P/\partial y \neq \partial Q/\partial x$ ,  $\mathbf{F}$  is not conservative by Theorem 5.

**V EXAMPLE 3** Determine whether or not the vector field

$$\mathbf{F}(x, y) = (3 + 2xy) \mathbf{i} + (x^2 - 3y^2) \mathbf{j}$$

is conservative.

**SOLUTION** Let  $P(x, y) = 3 + 2xy$  and  $Q(x, y) = x^2 - 3y^2$ . Then

$$\frac{\partial P}{\partial y} = 2x = \frac{\partial Q}{\partial x}$$

Also, the domain of  $\mathbf{F}$  is the entire plane ( $D = \mathbb{R}^2$ ), which is open and simply-connected. Therefore we can apply Theorem 6 and conclude that  $\mathbf{F}$  is conservative.

In Example 3, Theorem 6 told us that  $\mathbf{F}$  is conservative, but it did not tell us how to find the (potential) function  $f$  such that  $\mathbf{F} = \nabla f$ . The proof of Theorem 4 gives us a clue as to how to find  $f$ . We use “partial integration” as in the following example.

**EXAMPLE 4**

- (a) If  $\mathbf{F}(x, y) = (3 + 2xy) \mathbf{i} + (x^2 - 3y^2) \mathbf{j}$ , find a function  $f$  such that  $\mathbf{F} = \nabla f$ .  
 (b) Evaluate the line integral  $\int_C \mathbf{F} \cdot d\mathbf{r}$ , where  $C$  is the curve given by

$$\mathbf{r}(t) = e^t \sin t \mathbf{i} + e^t \cos t \mathbf{j} \quad 0 \leq t \leq \pi$$

**SOLUTION**

- (a) From Example 3 we know that  $\mathbf{F}$  is conservative and so there exists a function  $f$  with  $\nabla f = \mathbf{F}$ , that is,

$$f_x(x, y) = 3 + 2xy$$

$$f_y(x, y) = x^2 - 3y^2$$

Integrating [7] with respect to  $x$ , we obtain

$$\boxed{9} \quad f(x, y) = 3x + x^2y + g(y)$$

Notice that the constant of integration is a constant with respect to  $x$ , that is, a function of  $y$ , which we have called  $g(y)$ . Next we differentiate both sides of [9] with respect to  $y$ :

$$\boxed{10} \quad f_y(x, y) = x^2 + g'(y)$$

Comparing [8] and [10], we see that

$$g'(y) = -3y^2$$

Integrating with respect to  $y$ , we have

$$g(y) = -y^3 + K$$

where  $K$  is a constant. Putting this in [9], we have

$$f(x, y) = 3x + x^2y - y^3 + K$$

as the desired potential function.

(b) To use Theorem 2 all we have to know are the initial and terminal points of  $C$ , namely,  $\mathbf{r}(0) = (0, 1)$  and  $\mathbf{r}(\pi) = (0, -e^\pi)$ . In the expression for  $f(x, y)$  in part (a), any value of the constant  $K$  will do, so let's choose  $K = 0$ . Then we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(0, -e^\pi) - f(0, 1) = e^{3\pi} - (-1) = e^{3\pi} + 1$$

This method is much shorter than the straightforward method for evaluating line integrals that we learned in Section 16.2.  

A criterion for determining whether or not a vector field  $\mathbf{F}$  on  $\mathbb{R}^3$  is conservative is given in Section 16.5. Meanwhile, the next example shows that the technique for finding the potential function is much the same as for vector fields on  $\mathbb{R}^2$ .

**V EXAMPLE 5** If  $\mathbf{F}(x, y, z) = y^2 \mathbf{i} + (2xy + e^{3z}) \mathbf{j} + 3ye^{3z} \mathbf{k}$ , find a function  $f$  such that  $\nabla f = \mathbf{F}$ .

**SOLUTION** If there is such a function  $f$ , then

$$\boxed{11} \quad f_x(x, y, z) = y^2$$

$$\boxed{12} \quad f_y(x, y, z) = 2xy + e^{3z}$$

$$\boxed{13} \quad f_z(x, y, z) = 3ye^{3z}$$

Integrating [11] with respect to  $x$ , we get

$$\boxed{14} \quad f(x, y, z) = xy^2 + g(y, z)$$

where  $g(y, z)$  is a constant with respect to  $x$ . Then differentiating [14] with respect to  $y$ , we have

$$f_y(x, y, z) = 2xy + g_y(y, z)$$

and comparison with [12] gives

$$g_y(y, z) = e^{3z}$$

Thus  $g(y, z) = ye^{3z} + h(z)$  and we rewrite [14] as

$$f(x, y, z) = xy^2 + ye^{3z} + h(z)$$

Finally, differentiating with respect to  $z$  and comparing with [13], we obtain  $h'(z) = 0$  and therefore  $h(z) = K$ , a constant. The desired function is

$$f(x, y, z) = xy^2 + ye^{3z} + K$$

It is easily verified that  $\nabla f = \mathbf{F}$ . ■

### Conservation of Energy

Let's apply the ideas of this chapter to a continuous force field  $\mathbf{F}$  that moves an object along a path  $C$  given by  $\mathbf{r}(t)$ ,  $a \leq t \leq b$ , where  $\mathbf{r}(a) = A$  is the initial point and  $\mathbf{r}(b) = B$  is the terminal point of  $C$ . According to Newton's Second Law of Motion (see Section 13.4), the force  $\mathbf{F}(\mathbf{r}(t))$  at a point on  $C$  is related to the acceleration  $\mathbf{a}(t) = \mathbf{r}''(t)$  by the equation

$$\mathbf{F}(\mathbf{r}(t)) = m\mathbf{r}''(t)$$

So the work done by the force on the object is

$$\begin{aligned} W &= \int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt = \int_a^b m\mathbf{r}''(t) \cdot \mathbf{r}'(t) dt \\ &= \frac{m}{2} \int_a^b \frac{d}{dt} [\mathbf{r}'(t) \cdot \mathbf{r}'(t)] dt && \text{(Theorem 13.2.3, Formula 4)} \\ &= \frac{m}{2} \int_a^b \frac{d}{dt} |\mathbf{r}'(t)|^2 dt = \frac{m}{2} [|\mathbf{r}'(t)|^2]_a^b && \text{(Fundamental Theorem of Calculus)} \\ &= \frac{m}{2} (|\mathbf{r}'(b)|^2 - |\mathbf{r}'(a)|^2) \end{aligned}$$

Therefore

$$\boxed{15} \quad W = \frac{1}{2}m |\mathbf{v}(b)|^2 - \frac{1}{2}m |\mathbf{v}(a)|^2$$

where  $\mathbf{v} = \mathbf{r}'$  is the velocity.

The quantity  $\frac{1}{2}m |\mathbf{v}(t)|^2$ , that is, half the mass times the square of the speed, is called the **kinetic energy** of the object. Therefore we can rewrite Equation 15 as

$$\boxed{16} \quad W = K(B) - K(A)$$

which says that the work done by the force field along  $C$  is equal to the change in kinetic energy at the endpoints of  $C$ .

Now let's further assume that  $\mathbf{F}$  is a conservative force field; that is, we can write  $\mathbf{F} = \nabla f$ . In physics, the **potential energy** of an object at the point  $(x, y, z)$  is defined as  $P(x, y, z) = -f(x, y, z)$ , so we have  $\mathbf{F} = -\nabla P$ . Then by Theorem 2 we have

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = -\int_C \nabla P \cdot d\mathbf{r} = -[P(\mathbf{r}(b)) - P(\mathbf{r}(a))] = P(A) - P(B)$$

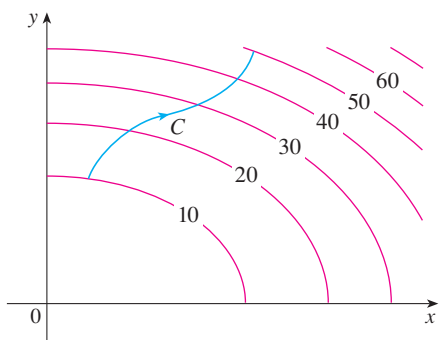
Comparing this equation with Equation 16, we see that

$$P(A) + K(A) = P(B) + K(B)$$

which says that if an object moves from one point  $A$  to another point  $B$  under the influence of a conservative force field, then the sum of its potential energy and its kinetic energy remains constant. This is called the **Law of Conservation of Energy** and it is the reason the vector field is called *conservative*.

### 16.3 Exercises

1. The figure shows a curve  $C$  and a contour map of a function  $f$  whose gradient is continuous. Find  $\int_C \nabla f \cdot d\mathbf{r}$ .



2. A table of values of a function  $f$  with continuous gradient is given. Find  $\int_C \nabla f \cdot d\mathbf{r}$ , where  $C$  has parametric equations

$$x = t^2 + 1 \quad y = t^3 + t \quad 0 \leq t \leq 1$$

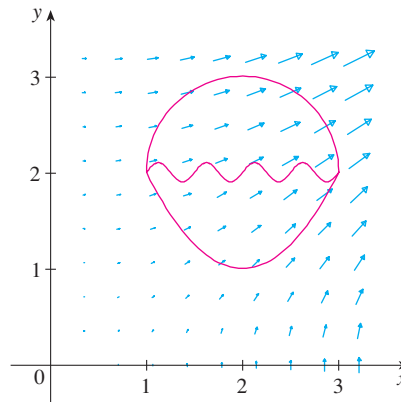
$x \backslash y$	0	1	2
0	1	6	4
1	3	5	7
2	8	2	9

- 3–10 Determine whether or not  $\mathbf{F}$  is a conservative vector field. If it is, find a function  $f$  such that  $\mathbf{F} = \nabla f$ .

3.  $\mathbf{F}(x, y) = (2x - 3y)\mathbf{i} + (-3x + 4y - 8)\mathbf{j}$
4.  $\mathbf{F}(x, y) = e^x \sin y \mathbf{i} + e^x \cos y \mathbf{j}$
5.  $\mathbf{F}(x, y) = e^x \cos y \mathbf{i} + e^x \sin y \mathbf{j}$
6.  $\mathbf{F}(x, y) = (3x^2 - 2y^2)\mathbf{i} + (4xy + 3)\mathbf{j}$
7.  $\mathbf{F}(x, y) = (ye^x + \sin y)\mathbf{i} + (e^x + x \cos y)\mathbf{j}$
8.  $\mathbf{F}(x, y) = (2xy + y^{-2})\mathbf{i} + (x^2 - 2xy^{-3})\mathbf{j}, \quad y > 0$

9.  $\mathbf{F}(x, y) = (\ln y + 2xy^3)\mathbf{i} + (3x^2y^2 + x/y)\mathbf{j}$
10.  $\mathbf{F}(x, y) = (xy \cosh xy + \sinh xy)\mathbf{i} + (x^2 \cosh xy)\mathbf{j}$

11. The figure shows the vector field  $\mathbf{F}(x, y) = \langle 2xy, x^2 \rangle$  and three curves that start at  $(1, 2)$  and end at  $(3, 2)$ .
- (a) Explain why  $\int_C \mathbf{F} \cdot d\mathbf{r}$  has the same value for all three curves.
  - (b) What is this common value?



- 12–18 (a) Find a function  $f$  such that  $\mathbf{F} = \nabla f$  and (b) use part (a) to evaluate  $\int_C \mathbf{F} \cdot d\mathbf{r}$  along the given curve  $C$ .

12.  $\mathbf{F}(x, y) = x^2\mathbf{i} + y^2\mathbf{j}$ ,  
 $C$  is the arc of the parabola  $y = 2x^2$  from  $(-1, 2)$  to  $(2, 8)$
13.  $\mathbf{F}(x, y) = xy^2\mathbf{i} + x^2y\mathbf{j}$ ,  
 $C: \mathbf{r}(t) = \langle t + \sin \frac{1}{2}\pi t, t + \cos \frac{1}{2}\pi t \rangle, \quad 0 \leq t \leq 1$
14.  $\mathbf{F}(x, y) = (1 + xy)e^{xy}\mathbf{i} + x^2e^{xy}\mathbf{j}$ ,  
 $C: \mathbf{r}(t) = \cos t \mathbf{i} + 2 \sin t \mathbf{j}, \quad 0 \leq t \leq \pi/2$
15.  $\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + (xy + 2z)\mathbf{k}$ ,  
 $C$  is the line segment from  $(1, 0, -2)$  to  $(4, 6, 3)$

16.  $\mathbf{F}(x, y, z) = (y^2z + 2xz^2)\mathbf{i} + 2xyz\mathbf{j} + (xy^2 + 2x^2z)\mathbf{k}$ ,  
 $C: x = \sqrt{t}, y = t + 1, z = t^2, 0 \leq t \leq 1$
17.  $\mathbf{F}(x, y, z) = yze^{xz}\mathbf{i} + e^{xz}\mathbf{j} + xye^{xz}\mathbf{k}$ ,  
 $C: \mathbf{r}(t) = (t^2 + 1)\mathbf{i} + (t^2 - 1)\mathbf{j} + (t^2 - 2t)\mathbf{k}, 0 \leq t \leq 2$
18.  $\mathbf{F}(x, y, z) = \sin y\mathbf{i} + (x \cos y + \cos z)\mathbf{j} - y \sin z\mathbf{k}$ ,  
 $C: \mathbf{r}(t) = \sin t\mathbf{i} + t\mathbf{j} + 2t\mathbf{k}, 0 \leq t \leq \pi/2$

19–20 Show that the line integral is independent of path and evaluate the integral.

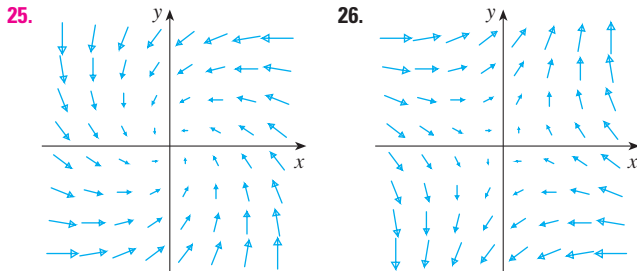
19.  $\int_C 2xe^{-y} dx + (2y - x^2e^{-y}) dy$ ,  
 $C$  is any path from  $(1, 0)$  to  $(2, 1)$
20.  $\int_C \sin y dx + (x \cos y - \sin y) dy$ ,  
 $C$  is any path from  $(2, 0)$  to  $(1, \pi)$

21. Suppose you're asked to determine the curve that requires the least work for a force field  $\mathbf{F}$  to move a particle from one point to another point. You decide to check first whether  $\mathbf{F}$  is conservative, and indeed it turns out that it is. How would you reply to the request?
22. Suppose an experiment determines that the amount of work required for a force field  $\mathbf{F}$  to move a particle from the point  $(1, 2)$  to the point  $(5, -3)$  along a curve  $C_1$  is 1.2 J and the work done by  $\mathbf{F}$  in moving the particle along another curve  $C_2$  between the same two points is 1.4 J. What can you say about  $\mathbf{F}$ ? Why?

23–24 Find the work done by the force field  $\mathbf{F}$  in moving an object from  $P$  to  $Q$ .

23.  $\mathbf{F}(x, y) = 2y^{3/2}\mathbf{i} + 3x\sqrt{y}\mathbf{j}; P(1, 1), Q(2, 4)$
24.  $\mathbf{F}(x, y) = e^{-y}\mathbf{i} - xe^{-y}\mathbf{j}; P(0, 1), Q(2, 0)$

25–26 Is the vector field shown in the figure conservative? Explain.



- CAS** 27. If  $\mathbf{F}(x, y) = \sin y\mathbf{i} + (1 + x \cos y)\mathbf{j}$ , use a plot to guess whether  $\mathbf{F}$  is conservative. Then determine whether your guess is correct.

28. Let  $\mathbf{F} = \nabla f$ , where  $f(x, y) = \sin(x - 2y)$ . Find curves  $C_1$  and  $C_2$  that are not closed and satisfy the equation.
- (a)  $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = 0$       (b)  $\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = 1$

29. Show that if the vector field  $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$  is conservative and  $P, Q, R$  have continuous first-order partial derivatives, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$$

30. Use Exercise 29 to show that the line integral  $\int_C y dx + x dy + xyz dz$  is not independent of path.

31–34 Determine whether or not the given set is (a) open, (b) connected, and (c) simply-connected.

31.  $\{(x, y) \mid 0 < y < 3\}$       32.  $\{(x, y) \mid 1 < |x| < 2\}$
33.  $\{(x, y) \mid 1 \leq x^2 + y^2 \leq 4, y \geq 0\}$
34.  $\{(x, y) \mid (x, y) \neq (2, 3)\}$

35. Let  $\mathbf{F}(x, y) = \frac{-y\mathbf{i} + x\mathbf{j}}{x^2 + y^2}$ .
- (a) Show that  $\partial P/\partial y = \partial Q/\partial x$ .
- (b) Show that  $\int_C \mathbf{F} \cdot d\mathbf{r}$  is not independent of path. [Hint: Compute  $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$  and  $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ , where  $C_1$  and  $C_2$  are the upper and lower halves of the circle  $x^2 + y^2 = 1$  from  $(1, 0)$  to  $(-1, 0)$ .] Does this contradict Theorem 6?

36. (a) Suppose that  $\mathbf{F}$  is an inverse square force field, that is,

$$\mathbf{F}(\mathbf{r}) = \frac{c\mathbf{r}}{|\mathbf{r}|^3}$$

for some constant  $c$ , where  $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ . Find the work done by  $\mathbf{F}$  in moving an object from a point  $P_1$  along a path to a point  $P_2$  in terms of the distances  $d_1$  and  $d_2$  from these points to the origin.

- (b) An example of an inverse square field is the gravitational field  $\mathbf{F} = -(mMG)\mathbf{r}/|\mathbf{r}|^3$  discussed in Example 4 in Section 16.1. Use part (a) to find the work done by the gravitational field when the earth moves from aphelion (at a maximum distance of  $1.52 \times 10^8$  km from the sun) to perihelion (at a minimum distance of  $1.47 \times 10^8$  km). (Use the values  $m = 5.97 \times 10^{24}$  kg,  $M = 1.99 \times 10^{30}$  kg, and  $G = 6.67 \times 10^{-11}$  N·m<sup>2</sup>/kg<sup>2</sup>.)
- (c) Another example of an inverse square field is the electric force field  $\mathbf{F} = \varepsilon qQ\mathbf{r}/|\mathbf{r}|^3$  discussed in Example 5 in Section 16.1. Suppose that an electron with a charge of  $-1.6 \times 10^{-19}$  C is located at the origin. A positive unit charge is positioned a distance  $10^{-12}$  m from the electron and moves to a position half that distance from the electron. Use part (a) to find the work done by the electric force field. (Use the value  $\varepsilon = 8.985 \times 10^9$ .)

## 16.6 Exercises

1–2 Determine whether the points  $P$  and  $Q$  lie on the given surface.

1.  $\mathbf{r}(u, v) = \langle 2u + 3v, 1 + 5u - v, 2 + u + v \rangle$   
 $P(7, 10, 4), Q(5, 22, 5)$

2.  $\mathbf{r}(u, v) = \langle u + v, u^2 - v, u + v^2 \rangle$   
 $P(3, -1, 5), Q(-1, 3, 4)$


3–6 Identify the surface with the given vector equation.

3.  $\mathbf{r}(u, v) = (u + v)\mathbf{i} + (3 - v)\mathbf{j} + (1 + 4u + 5v)\mathbf{k}$

4.  $\mathbf{r}(u, v) = 2 \sin u \mathbf{i} + 3 \cos u \mathbf{j} + v \mathbf{k}, \quad 0 \leq v \leq 2$

5.  $\mathbf{r}(s, t) = \langle s, t, t^2 - s^2 \rangle$

6.  $\mathbf{r}(s, t) = \langle s \sin 2t, s^2, s \cos 2t \rangle$

 7–12 Use a computer to graph the parametric surface. Get a printout and indicate on it which grid curves have  $u$  constant and which have  $v$  constant.

7.  $\mathbf{r}(u, v) = \langle u^2, v^2, u + v \rangle,$   
 $-1 \leq u \leq 1, -1 \leq v \leq 1$

8.  $\mathbf{r}(u, v) = \langle u, v^3, -v \rangle,$   
 $-2 \leq u \leq 2, -2 \leq v \leq 2$

9.  $\mathbf{r}(u, v) = \langle u \cos v, u \sin v, u^5 \rangle,$   
 $-1 \leq u \leq 1, 0 \leq v \leq 2\pi$

10.  $\mathbf{r}(u, v) = \langle u, \sin(u + v), \sin v \rangle,$   
 $-\pi \leq u \leq \pi, -\pi \leq v \leq \pi$

11.  $x = \sin v, \quad y = \cos u \sin 4v, \quad z = \sin 2u \sin 4v,$   
 $0 \leq u \leq 2\pi, -\pi/2 \leq v \leq \pi/2$

12.  $x = \sin u, \quad y = \cos u \sin v, \quad z = \sin v,$   
 $0 \leq u \leq 2\pi, 0 \leq v \leq 2\pi$

13–18 Match the equations with the graphs labeled I–VI and give reasons for your answers. Determine which families of grid curves have  $u$  constant and which have  $v$  constant.

13.  $\mathbf{r}(u, v) = u \cos v \mathbf{i} + u \sin v \mathbf{j} + v \mathbf{k}$

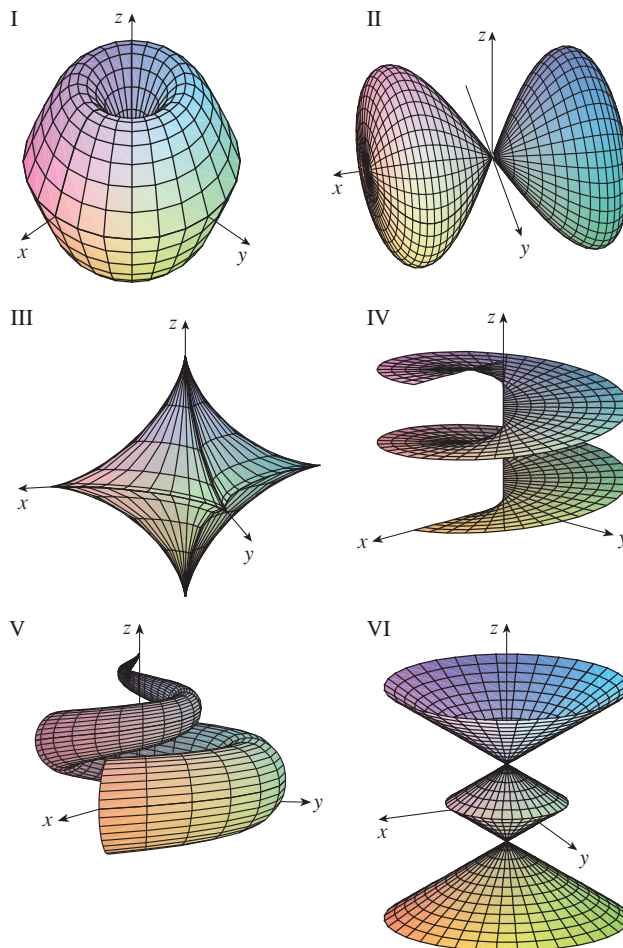
14.  $\mathbf{r}(u, v) = u \cos v \mathbf{i} + u \sin v \mathbf{j} + \sin u \mathbf{k}, \quad -\pi \leq u \leq \pi$

15.  $\mathbf{r}(u, v) = \sin v \mathbf{i} + \cos u \sin 2v \mathbf{j} + \sin u \sin 2v \mathbf{k}$

16.  $x = (1 - u)(3 + \cos v) \cos 4\pi u,$   
 $y = (1 - u)(3 + \cos v) \sin 4\pi u,$   
 $z = 3u + (1 - u) \sin v$

17.  $x = \cos^3 u \cos^3 v, \quad y = \sin^3 u \cos^3 v, \quad z = \sin^3 v$

18.  $x = (1 - |u|) \cos v, \quad y = (1 - |u|) \sin v, \quad z = u$



19–26 Find a parametric representation for the surface.

19. The plane through the origin that contains the vectors  $\mathbf{i} - \mathbf{j}$  and  $\mathbf{j} - \mathbf{k}$

20. The plane that passes through the point  $(0, -1, 5)$  and contains the vectors  $\langle 2, 1, 4 \rangle$  and  $\langle -3, 2, 5 \rangle$

21. The part of the hyperboloid  $4x^2 - 4y^2 - z^2 = 4$  that lies in front of the  $yz$ -plane

22. The part of the ellipsoid  $x^2 + 2y^2 + 3z^2 = 1$  that lies to the left of the  $xz$ -plane

23. The part of the sphere  $x^2 + y^2 + z^2 = 4$  that lies above the cone  $z = \sqrt{x^2 + y^2}$

24. The part of the sphere  $x^2 + y^2 + z^2 = 16$  that lies between the planes  $z = -2$  and  $z = 2$

25. The part of the cylinder  $y^2 + z^2 = 16$  that lies between the planes  $x = 0$  and  $x = 5$