59–60 Find the average value of f over the region D.

59. $f(x, y) = xy$, D is the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 3)$

60. $f(x, y) = x \sin y$, D is enclosed by the curves $y = 0$, $y = x^2$, and $x = 1$

61. Prove Property 11.

62. In evaluating a double integral over a region D, a sum of iterated integrals was obtained as follows:

$$\iint_D f(x, y)\, dA = \int_0^1 \int_0^{2y} f(x, y)\, dx\, dy + \int_1^3 \int_0^{3-y} f(x, y)\, dx\, dy$$

Sketch the region D and express the double integral as an iterated integral with reversed order of integration.

63–67 Use geometry or symmetry, or both, to evaluate the double integral.

63. $\displaystyle\iint_D (x + 2)\, dA$, $D = \{(x, y) \mid 0 \leqslant y \leqslant \sqrt{9 - x^2}\}$

64. $\displaystyle\iint_D \sqrt{R^2 - x^2 - y^2}\, dA$,
D is the disk with center the origin and radius R

65. $\displaystyle\iint_D (2x + 3y)\, dA$,
D is the rectangle $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant b$

66. $\displaystyle\iint_D (2 + x^2y^3 - y^2 \sin x)\, dA$,
$D = \{(x, y) \mid |x| + |y| \leqslant 1\}$

67. $\displaystyle\iint_D \left(ax^3 + by^3 + \sqrt{a^2 - x^2}\right) dA$,
$D = [-a, a] \times [-b, b]$

CAS **68.** Graph the solid bounded by the plane $x + y + z = 1$ and the paraboloid $z = 4 - x^2 - y^2$ and find its exact volume. (Use your CAS to do the graphing, to find the equations of the boundary curves of the region of integration, and to evaluate the double integral.)

15.4 Double Integrals in Polar Coordinates

Suppose that we want to evaluate a double integral $\iint_R f(x, y)\, dA$, where R is one of the regions shown in Figure 1. In either case the description of R in terms of rectangular coordinates is rather complicated, but R is easily described using polar coordinates.

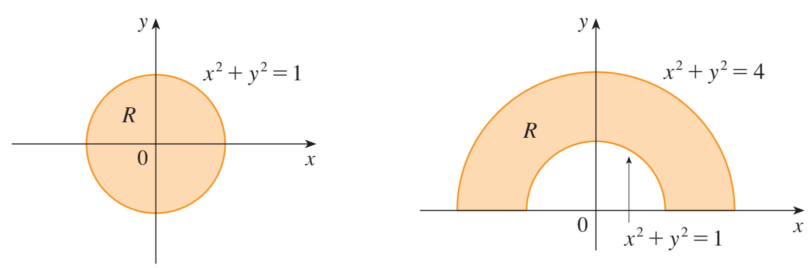


FIGURE 1

(a) $R = \{(r, \theta) \mid 0 \leqslant r \leqslant 1, 0 \leqslant \theta \leqslant 2\pi\}$ (b) $R = \{(r, \theta) \mid 1 \leqslant r \leqslant 2, 0 \leqslant \theta \leqslant \pi\}$

Recall from Figure 2 that the polar coordinates (r, θ) of a point are related to the rectangular coordinates (x, y) by the equations

$$\boxed{r^2 = x^2 + y^2 \qquad x = r\cos\theta \qquad y = r\sin\theta}$$

(See Section 10.3.)

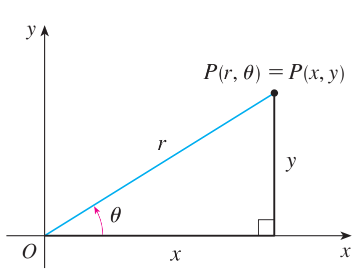


FIGURE 2

The regions in Figure 1 are special cases of a **polar rectangle**

$$R = \{(r, \theta) \mid a \leqslant r \leqslant b, \alpha \leqslant \theta \leqslant \beta\}$$

which is shown in Figure 3. In order to compute the double integral $\iint_R f(x, y)\, dA$, where R is a polar rectangle, we divide the interval $[a, b]$ into m subintervals $[r_{i-1}, r_i]$ of equal width $\Delta r = (b - a)/m$ and we divide the interval $[\alpha, \beta]$ into n subintervals $[\theta_{j-1}, \theta_j]$ of equal width $\Delta\theta = (\beta - \alpha)/n$. Then the circles $r = r_i$ and the rays $\theta = \theta_j$ divide the polar rectangle R into the small polar rectangles R_{ij} shown in Figure 4.

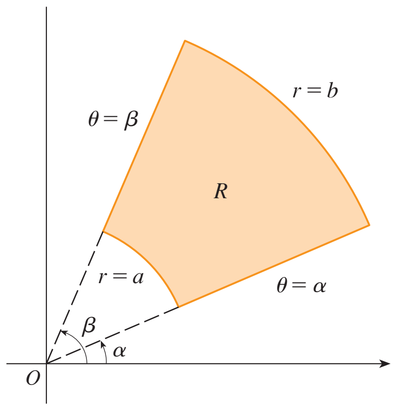


FIGURE 3 Polar rectangle

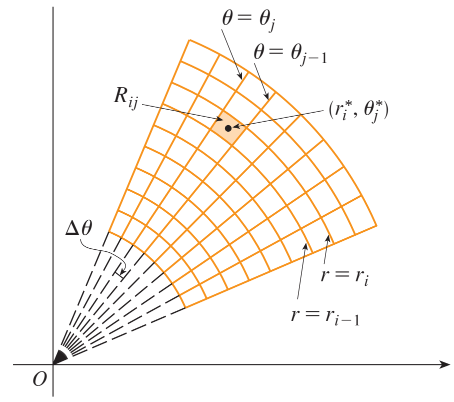


FIGURE 4 Dividing R into polar subrectangles

The "center" of the polar subrectangle

$$R_{ij} = \{(r, \theta) \mid r_{i-1} \leqslant r \leqslant r_i,\ \theta_{j-1} \leqslant \theta \leqslant \theta_j\}$$

has polar coordinates

$$r_i^* = \tfrac{1}{2}(r_{i-1} + r_i) \qquad \theta_j^* = \tfrac{1}{2}(\theta_{j-1} + \theta_j)$$

We compute the area of R_{ij} using the fact that the area of a sector of a circle with radius r and central angle θ is $\frac{1}{2}r^2\theta$. Subtracting the areas of two such sectors, each of which has central angle $\Delta\theta = \theta_j - \theta_{j-1}$, we find that the area of R_{ij} is

$$\begin{aligned} \Delta A_i &= \tfrac{1}{2}r_i^2\,\Delta\theta - \tfrac{1}{2}r_{i-1}^2\,\Delta\theta = \tfrac{1}{2}(r_i^2 - r_{i-1}^2)\,\Delta\theta \\ &= \tfrac{1}{2}(r_i + r_{i-1})(r_i - r_{i-1})\,\Delta\theta = r_i^*\,\Delta r\,\Delta\theta \end{aligned}$$

Although we have defined the double integral $\iint_R f(x, y)\, dA$ in terms of ordinary rectangles, it can be shown that, for continuous functions f, we always obtain the same answer using polar rectangles. The rectangular coordinates of the center of R_{ij} are $(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*)$, so a typical Riemann sum is

$$\boxed{1} \quad \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*)\,\Delta A_i = \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*)\, r_i^*\,\Delta r\,\Delta\theta$$

If we write $g(r, \theta) = r f(r\cos\theta, r\sin\theta)$, then the Riemann sum in Equation 1 can be written as

$$\sum_{i=1}^{m}\sum_{j=1}^{n} g(r_i^*, \theta_j^*)\,\Delta r\,\Delta\theta$$

which is a Riemann sum for the double integral

$$\int_\alpha^\beta \int_a^b g(r, \theta)\, dr\, d\theta$$

Therefore we have

$$\begin{aligned}\iint_R f(x, y)\, dA &= \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(r_i^* \cos \theta_j^*, r_i^* \sin \theta_j^*)\, \Delta A_i \\ &= \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} g(r_i^*, \theta_j^*)\, \Delta r\, \Delta\theta = \int_\alpha^\beta \int_a^b g(r, \theta)\, dr\, d\theta \\ &= \int_\alpha^\beta \int_a^b f(r \cos \theta, r \sin \theta)\, r\, dr\, d\theta\end{aligned}$$

2 Change to Polar Coordinates in a Double Integral If f is continuous on a polar rectangle R given by $0 \leqslant a \leqslant r \leqslant b$, $\alpha \leqslant \theta \leqslant \beta$, where $0 \leqslant \beta - \alpha \leqslant 2\pi$, then

$$\iint_R f(x, y)\, dA = \int_\alpha^\beta \int_a^b f(r \cos \theta, r \sin \theta)\, r\, dr\, d\theta$$

The formula in 2 says that we convert from rectangular to polar coordinates in a double integral by writing $x = r \cos \theta$ and $y = r \sin \theta$, using the appropriate limits of integration for r and θ, and replacing dA by $r\, dr\, d\theta$. Be careful not to forget the additional factor r on the right side of Formula 2. A classical method for remembering this is shown in Figure 5, where the "infinitesimal" polar rectangle can be thought of as an ordinary rectangle with dimensions $r\, d\theta$ and dr and therefore has "area" $dA = r\, dr\, d\theta$.

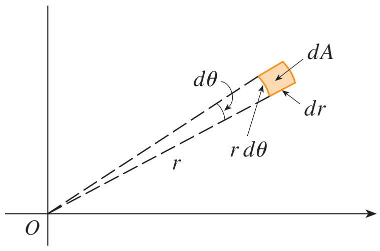


FIGURE 5

EXAMPLE 1 Evaluate $\iint_R (3x + 4y^2)\, dA$, where R is the region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

SOLUTION The region R can be described as

$$R = \{(x, y) \mid y \geqslant 0,\ 1 \leqslant x^2 + y^2 \leqslant 4\}$$

It is the half-ring shown in Figure 1(b), and in polar coordinates it is given by $1 \leqslant r \leqslant 2$, $0 \leqslant \theta \leqslant \pi$. Therefore, by Formula 2,

$$\begin{aligned}\iint_R (3x + 4y^2)\, dA &= \int_0^\pi \int_1^2 (3r \cos \theta + 4r^2 \sin^2\theta)\, r\, dr\, d\theta \\ &= \int_0^\pi \int_1^2 (3r^2 \cos \theta + 4r^3 \sin^2\theta)\, dr\, d\theta \\ &= \int_0^\pi \Big[r^3 \cos \theta + r^4 \sin^2\theta\Big]_{r=1}^{r=2} d\theta = \int_0^\pi (7 \cos \theta + 15 \sin^2\theta)\, d\theta \\ &= \int_0^\pi \left[7 \cos \theta + \tfrac{15}{2}(1 - \cos 2\theta)\right] d\theta \\ &= 7 \sin \theta + \frac{15\theta}{2} - \frac{15}{4} \sin 2\theta \Bigg]_0^\pi = \frac{15\pi}{2}\end{aligned}$$

Here we use the trigonometric identity

$$\sin^2\theta = \tfrac{1}{2}(1 - \cos 2\theta)$$

See Section 7.2 for advice on integrating trigonometric functions.

V EXAMPLE 2 Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$.

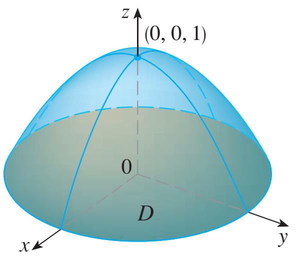


FIGURE 6

SOLUTION If we put $z = 0$ in the equation of the paraboloid, we get $x^2 + y^2 = 1$. This means that the plane intersects the paraboloid in the circle $x^2 + y^2 = 1$, so the solid lies under the paraboloid and above the circular disk D given by $x^2 + y^2 \leqslant 1$ [see Figures 6 and 1(a)]. In polar coordinates D is given by $0 \leqslant r \leqslant 1$, $0 \leqslant \theta \leqslant 2\pi$. Since $1 - x^2 - y^2 = 1 - r^2$, the volume is

$$V = \iint_D (1 - x^2 - y^2)\, dA = \int_0^{2\pi} \int_0^1 (1 - r^2)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^1 (r - r^3)\, dr = 2\pi \left[\frac{r^2}{2} - \frac{r^4}{4} \right]_0^1 = \frac{\pi}{2}$$

If we had used rectangular coordinates instead of polar coordinates, then we would have obtained

$$V = \iint_D (1 - x^2 - y^2)\, dA = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1 - x^2 - y^2)\, dy\, dx$$

which is not easy to evaluate because it involves finding $\int (1 - x^2)^{3/2}\, dx$.

What we have done so far can be extended to the more complicated type of region shown in Figure 7. It's similar to the type II rectangular regions considered in Section 15.3. In fact, by combining Formula 2 in this section with Formula 15.3.5, we obtain the following formula.

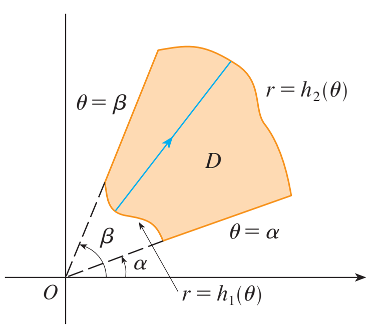


FIGURE 7
$D = \{(r, \theta) \mid \alpha \leqslant \theta \leqslant \beta,\ h_1(\theta) \leqslant r \leqslant h_2(\theta)\}$

3 If f is continuous on a polar region of the form

$$D = \{(r, \theta) \mid \alpha \leqslant \theta \leqslant \beta,\ h_1(\theta) \leqslant r \leqslant h_2(\theta)\}$$

then
$$\iint_D f(x, y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

In particular, taking $f(x, y) = 1$, $h_1(\theta) = 0$, and $h_2(\theta) = h(\theta)$ in this formula, we see that the area of the region D bounded by $\theta = \alpha$, $\theta = \beta$, and $r = h(\theta)$ is

$$A(D) = \iint_D 1\, dA = \int_\alpha^\beta \int_0^{h(\theta)} r\, dr\, d\theta$$

$$= \int_\alpha^\beta \left[\frac{r^2}{2} \right]_0^{h(\theta)} d\theta = \int_\alpha^\beta \tfrac{1}{2}[h(\theta)]^2\, d\theta$$

and this agrees with Formula 10.4.3.

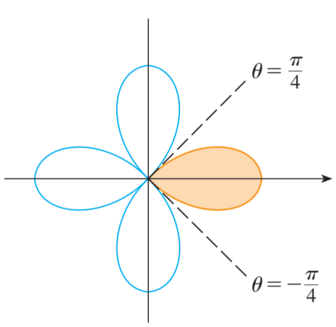


FIGURE 8

V EXAMPLE 3 Use a double integral to find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.

SOLUTION From the sketch of the curve in Figure 8, we see that a loop is given by the region

$$D = \{(r, \theta) \mid -\pi/4 \leqslant \theta \leqslant \pi/4,\ 0 \leqslant r \leqslant \cos 2\theta\}$$

So the area is

$$A(D) = \iint_D dA = \int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\theta} r\,dr\,d\theta$$

$$= \int_{-\pi/4}^{\pi/4} \left[\tfrac{1}{2}r^2\right]_0^{\cos 2\theta} d\theta = \tfrac{1}{2}\int_{-\pi/4}^{\pi/4} \cos^2 2\theta\,d\theta$$

$$= \tfrac{1}{4}\int_{-\pi/4}^{\pi/4} (1 + \cos 4\theta)\,d\theta = \tfrac{1}{4}\Big[\theta + \tfrac{1}{4}\sin 4\theta\Big]_{-\pi/4}^{\pi/4} = \frac{\pi}{8}$$

V EXAMPLE 4 Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 2x$.

SOLUTION The solid lies above the disk D whose boundary circle has equation $x^2 + y^2 = 2x$ or, after completing the square,

$$(x - 1)^2 + y^2 = 1$$

(See Figures 9 and 10.)

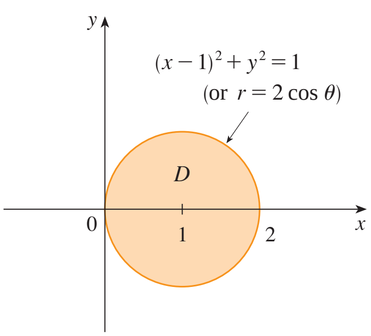


FIGURE 9

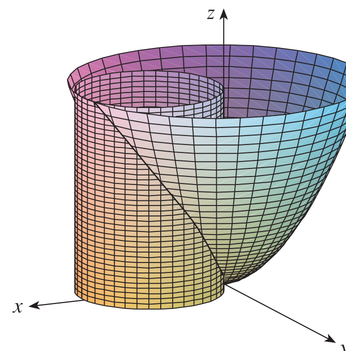


FIGURE 10

In polar coordinates we have $x^2 + y^2 = r^2$ and $x = r\cos\theta$, so the boundary circle becomes $r^2 = 2r\cos\theta$, or $r = 2\cos\theta$. Thus the disk D is given by

$$D = \{(r, \theta) \mid -\pi/2 \leqslant \theta \leqslant \pi/2,\ 0 \leqslant r \leqslant 2\cos\theta\}$$

and, by Formula 3, we have

$$V = \iint_D (x^2 + y^2)\,dA = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} r^2\,r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2} \left[\frac{r^4}{4}\right]_0^{2\cos\theta} d\theta$$

$$= 4\int_{-\pi/2}^{\pi/2} \cos^4\theta\,d\theta = 8\int_0^{\pi/2} \cos^4\theta\,d\theta = 8\int_0^{\pi/2} \left(\frac{1 + \cos 2\theta}{2}\right)^2 d\theta$$

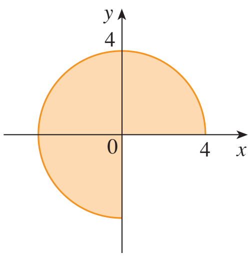
$$= 2\int_0^{\pi/2} \Big[1 + 2\cos 2\theta + \tfrac{1}{2}(1 + \cos 4\theta)\Big]\,d\theta$$

$$= 2\Big[\tfrac{3}{2}\theta + \sin 2\theta + \tfrac{1}{8}\sin 4\theta\Big]_0^{\pi/2} = 2\left(\frac{3}{2}\right)\left(\frac{\pi}{2}\right) = \frac{3\pi}{2}$$

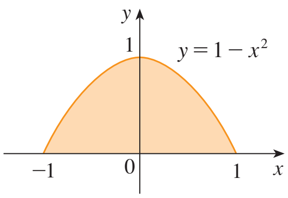
15.4 Exercises

1–4 A region R is shown. Decide whether to use polar coordinates or rectangular coordinates and write $\iint_R f(x, y)\, dA$ as an iterated integral, where f is an arbitrary continuous function on R.

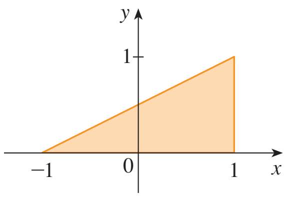
1.



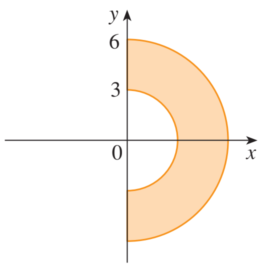
2.



3.



4.



5–6 Sketch the region whose area is given by the integral and evaluate the integral.

5. $\displaystyle\int_{\pi/4}^{3\pi/4} \int_1^2 r\, dr\, d\theta$ **6.** $\displaystyle\int_{\pi/2}^{\pi} \int_0^{2\sin\theta} r\, dr\, d\theta$

7–14 Evaluate the given integral by changing to polar coordinates.

7. $\iint_D x^2 y\, dA$, where D is the top half of the disk with center the origin and radius 5

8. $\iint_R (2x - y)\, dA$, where R is the region in the first quadrant enclosed by the circle $x^2 + y^2 = 4$ and the lines $x = 0$ and $y = x$

9. $\iint_R \sin(x^2 + y^2)\, dA$, where R is the region in the first quadrant between the circles with center the origin and radii 1 and 3

10. $\displaystyle\iint_R \frac{y^2}{x^2 + y^2}\, dA$, where R is the region that lies between the circles $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$ with $0 < a < b$

11. $\iint_D e^{-x^2-y^2}\, dA$, where D is the region bounded by the semicircle $x = \sqrt{4 - y^2}$ and the y-axis

12. $\iint_D \cos\sqrt{x^2 + y^2}\, dA$, where D is the disk with center the origin and radius 2

13. $\iint_R \arctan(y/x)\, dA$,
where $R = \{(x, y) \mid 1 \leqslant x^2 + y^2 \leqslant 4,\ 0 \leqslant y \leqslant x\}$

14. $\iint_D x\, dA$, where D is the region in the first quadrant that lies between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 2x$

15–18 Use a double integral to find the area of the region.

15. One loop of the rose $r = \cos 3\theta$

16. The region enclosed by both of the cardioids $r = 1 + \cos\theta$ and $r = 1 - \cos\theta$

17. The region inside the circle $(x - 1)^2 + y^2 = 1$ and outside the circle $x^2 + y^2 = 1$

18. The region inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 3\cos\theta$

19–27 Use polar coordinates to find the volume of the given solid.

19. Under the cone $z = \sqrt{x^2 + y^2}$ and above the disk $x^2 + y^2 \leqslant 4$

20. Below the paraboloid $z = 18 - 2x^2 - 2y^2$ and above the xy-plane

21. Enclosed by the hyperboloid $-x^2 - y^2 + z^2 = 1$ and the plane $z = 2$

22. Inside the sphere $x^2 + y^2 + z^2 = 16$ and outside the cylinder $x^2 + y^2 = 4$

23. A sphere of radius a

24. Bounded by the paraboloid $z = 1 + 2x^2 + 2y^2$ and the plane $z = 7$ in the first octant

25. Above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 1$

26. Bounded by the paraboloids $z = 3x^2 + 3y^2$ and $z = 4 - x^2 - y^2$

27. Inside both the cylinder $x^2 + y^2 = 4$ and the ellipsoid $4x^2 + 4y^2 + z^2 = 64$

28. (a) A cylindrical drill with radius r_1 is used to bore a hole through the center of a sphere of radius r_2. Find the volume of the ring-shaped solid that remains.
(b) Express the volume in part (a) in terms of the height h of the ring. Notice that the volume depends only on h, not on r_1 or r_2.

29–32 Evaluate the iterated integral by converting to polar coordinates.

29. $\displaystyle\int_{-3}^{3} \int_0^{\sqrt{9-x^2}} \sin(x^2 + y^2)\, dy\, dx$ **30.** $\displaystyle\int_0^a \int_{-\sqrt{a^2-y^2}}^{0} x^2 y\, dx\, dy$

31. $\displaystyle\int_0^1 \int_y^{\sqrt{2-y^2}} (x + y)\, dx\, dy$ **32.** $\displaystyle\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2 + y^2}\, dy\, dx$

1. Homework Hints available at stewartcalculus.com